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Sreoshi Banerjee
(Indian Statistical Institute, Calcutta)
Parikshit De
(Indian Institute of Science Education and Research, Bhopal)
Manipushpak Mitra
(Indian Statistical Institute, Calcutta)

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ACCEPTABLE UTILITY BOUNDS IN SEQUENCING PROBLEMS WITH INCENTIVES

SREOSHI BANERJEE, PARIKSHIT DE, AND MANIPUSHPAK MITRA

ABSTRACT. In a sequencing environment with incomplete information, we study the impact of imposing a lower bound on the utility function of agents. We call this the ”acceptable utility bound”. Such a bound guarantees a minimum acceptable utility to every agent and acts as a veil of protection. Our primary motive is to identify the class of outcome efficient and strategy-proof mechanisms which satisfy the ”acceptable utility bound”. We identify a necessary and sufficient condition to obtain such a class of mechanisms. This is followed by our characterization result where the set of mechanisms satisfying outcome efficiency, strategy-proofness and the acceptable utility bound are termed as ”relative pivotal mechanisms”. The paper also provides relevant theoretical applications involving specific lower bounds namely; bounds with initial order, identical cost bounds and expected cost bounds. We also offer insights on the issue of feasibility and/or budget balance.

JEL Classifications: C72, D63, D71, D82;

Keywords: sequencing problems, acceptable utility bounds, outcome efficiency, strategyproofness, feasibility, budget balance.

1. INTRODUCTION

1.1. **Purpose.** In a private information set up, we consider a mechanism design problem to study the implications of providing a guaranteed level of utility to every agent when monetary transfers are allowed. A lower bound on an individual's utility, acts as a safety net, as and when his final welfare is realized. An

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extremely crucial and concrete example highlighting the relevance of such a guarantee can be found in the health care sector. In Sweden, long waiting lines for surgical procedures pose a threat to the quality of their health policy agenda. To reduce waiting lists, in 1992 the Swedish Government and the Federation of County Council agreed on an initiative to offer a maximum waiting-time guarantee. Patients awaiting medical procedures are guaranteed a waiting time no longer than 3 months from the physician's decision to treat/operate (see Hanning [22]). Similarly, UK's national health service (NHS) provides emergency patients with a four hours target window within which 95 percent of the patients need to be discharged or transferred [1].

In our everyday lives, we hear such assurances at crowded restaurants where the manager declares a minimum estimated waiting time or at pizza home delivery services where a customer is provided a serial number and guaranteed delivery within 30 minutes, failing which the order is served free of cost. Another crucial instance involves the massive congestion of vehicles at toll plazas pan India. The National Highway Authority of India (NHAI) ensures that the number of toll lanes/booths are such that, the service time per vehicle during peak hours is not more than 10 seconds. The NHAI rules also suggest an increase in the number of toll lanes if the waiting time of the users exceeds 3 minutes. Moreover, there are specific regions in the country where riders are exempted from paying the toll tax altogether if the total waiting time surpasses 3 minutes. All these real life scenarios indicate the necessity of setting benchmark thresholds to smoothen customer experience and in turn, guarantee a minimum level of utility to the public at large.

1.2. **Our framework.** We work in a standard sequencing environment with a finite set of agents. Long waiting lines are a terrifying phenomenon in a world where everyone needs everything in an instant. Queue management at supermarkets, airport check-in counters, hospitals (especially outpatient department), toll plazas, railway ticket counters, etc is vital to customers and service providers. In our model, each agent has a single job to process using a facility that can only

[1] https://www.nhsinform.scot/care-support-and-rights/health-rights/access/waiting-times

serve one agent's requirement at a time. It is assumed that no job can be interrupted once it starts processing. A job is characterized by its processing time and an agent's waiting cost. The latter represents the disutility of waiting (per unit of time). The processing time of all agents are publicly known while the waiting costs are private information.

The sequencing model broadly captures a very realistic and persistent economic problem that hampers customer satisfaction and long term loyalty. Business organizations around the world are trying to adopt strategies to manage queues better and improve overall consumer experience through monetary and non-monetary incentives. Theme parks optimize long queues by offering chargeable express passes during peak hours and special discounts for guests arriving late in the evening when the crowd thins out. The Disneyland queue management team entertains patrons waiting in long lines through dance, music, character parades and other attractions. Airports often try to ease congestion and reduce customer waiting time by offering priority check-ins to their passengers at a very nominal fee. Amazon guarantees faster delivery to their "prime" members in lieu of an annual payment and offers a cashback to those customers who are willing to wait longer for their product delivery. Our model assumes that agents have quasi-linear preferences and the mechanism designer allows for monetary incentives. There is a well established literature in this direction.[2] A special case of sequencing problems where the processing times of the agents are identical is called queueing problems. Queueing problems have also been analyzed extensively from both normative and strategic viewpoints.[3]

Our paper imposes, what is termed as the "acceptable utility bound", on the utility function of agents. In a sequencing problem, this lower bound offers a veil of protection to every agent against the maximum dis-utility incurred. One must note that, such a bound differs from the concept of a participation constraint, which is an outside option guarantee, assuring that agents who participate will

[2] See De [13], [14], De and Mitra [15], [16], Dolan [17], Duives, Heydenreich, Mishra, Muller and Uetz [18], Hain and Mitra [21], Mitra [28], Moulin [31] and Suijs [37].

[3] See Chun [2], [3], Chun, Mitra and Mutuswami [5], [6], [7], [8], Hashimoto and Saitoh [23], Kayi and Ramaekars [25], Maniquet [26], Mitra [27], [29], Mitra and Mutuswami [30] and Mukherjee [34].

remain at least as well off as they would have been if they hadn't participated. An acceptable utility bound is endogenously constructed once the set of agents who are participating, is determined and known to the mechanism designer. The literature has analysed the impact of specific lower bounds on sequencing and queueing models. These bounds have been separately elaborated in our application section. Our analysis is based on a general representation capturing all such bounds, thus providing a standard platform for further research. The acceptable utility bound has two components namely; the waiting cost of an agent and the acceptability parameter. The acceptability parameter is a functional form of the job processing time of an agent and varies depending on the bound under consideration. Due to the linearity of the cost structure in our model, the acceptable utility bound is taken to be the product of these two components.

A sequencing rule is outcome efficient if it minimizes the aggregate job completion cost. A mechanism implements a sequencing rule in dominant strategies if the transfer is such that truthful reporting for any agent weakly dominates false reporting irrespective of what other agents declare. Implementation of outcome efficient sequencing rules in dominant strategies has been well studied in the literature on mechanism design under incomplete information. It is also well-known that, as long as preferences are 'smoothly connected' (see Holmström [24]), outcome efficient rules can be implemented in dominant strategies if and only if the mechanism is a Vickrey-Clarke-Groves (VCG) mechanism (see Clarke [9], Groves [19] and Vickrey [38]). For sequencing problems, mechanism design under incomplete information was analyzed by Dolan [17], Hain and Mitra [21], Moulin [31], Mitra [28] and Suijs [37].

1.3. **Results.** In our first result, we identify the "constrained acceptability property" which is a condition that is both necessary and sufficient to obtain outcome efficient and strategyproof mechanisms that satisfy the acceptable utility bound. Constrained acceptability property requires that every agent's acceptability parameter must be bounded below by his job completion time when he occupies the first position in the queue.

Given this property, our second theorem is a characterization result where we introduce the class of 'relative pivotal mechanisms' which is a strict subset of the set of all VCG mechanisms and satisfy the acceptable utility bound. For any given vector of waiting costs, the main aspect of a relative pivotal mechanism is to construct a 'benchmark' waiting cost. This is based on an optimization exercise conducted using the acceptability parameter of the agent and waiting costs of all other agents. Given the benchmark waiting costs of all agents, under the relative pivotal mechanism, the transfer of each agent has three parts. One part of the transfer depends on the difference between his acceptability parameter and his cost with this benchmark waiting cost. The other part of the transfer is based on the calculation of externality caused by this agent with his waiting cost on all other agents in comparison to what would have happened if, ceteris paribus, this agent had the benchmark waiting cost. The third part of the transfer is any non-negative valued function that depends on the waiting cost of all other agents.

Next we address the issue of finding relative pivotal mechanisms that satisfy either feasibility or its stronger version called budget balance.[4] We begin by identifying the "weighted net acceptability" property which is a necessary condition to find mechanisms satisfying acceptable utility bounds, outcome efficiency and feasibility. We show that when there are two agents we can only get feasible (and not budget balanced) relative pivotal mechanisms if and only if each agent's acceptability parameter equals the cost associated with getting served last. For more than two agents we can show that if the acceptability parameter of each agent is the cost associated with getting served last, then we can get budget balanced (hence, feasible) relative pivotal mechanism.

1.4. **Applications.** We apply our general results to sequencing problems with a natural ex-ante initial order (most commonly observed in our day to day lives). Our next application captures the essence of fairness by constructing an egalitarian bound that treats agents identically such that no one agent suffers due to the heterogeneity of other's preferences. In our final application, we allow for random arrival of queues. In other words, every possible ordering of agents has an equal chance of arriving to avail a service.

[4]It is well-known that feasibility of a mechanism requires that the sum of transfers across all agents is non-positive and budget balance requires that the sum of transfers across all agents is zero.

For sequencing problems with initial order, there is a preexisting order on the agents. From the cooperative game perspective, this type of sequencing problem with initial order was analyzed for sequencing games by Curiel, Pederzoli, Tijs [12] and, for queueing problems with a mechanism design perspective, this problem was addressed by Chun, Mitra and Mutuswami [7] and by Gershkov and Schweinzer [20]. Any sequencing problems with a given initial order satisfies the constrained acceptability property. Hence, for sequencing problems with initial order, achieving outcome efficiency and eliciting private information boils down to reordering the existing initial order to the outcome efficient order by using relative pivotal mechanisms. In this context we can show that there is no feasible (and hence no budget balanced) relative pivotal mechanism.[5]

In the next set up, we apply two notions of acceptable utility bounds-the identical cost bounds (ICB) and the expected cost bounds (ECB). ICB is a well-known notion of fairness that has been used in many contexts.[6] ICB requires that each agent receives at least the utility he could expect under the egalitarian solution if all agents were like him in a reference economy. The reference economy for any agent i requires that all other agents have the same waiting cost and processing time as agent i. Since agents are identical in this sense, each of them has an equal right to the resource. As a consequence agent i faces all possible orders of serving the agents with equal chance. For queueing problems, the notion of ICB was analyzed by Chun and Yengin [11], Kayi and Ramaekers [25] and Mitra [29]. In the queueing context, Gershkov and Schweinzer [20] considered the random arrival rescheduling problem that generates another type of acceptable utility bounds which we call the "expected cost bounds" (ECB). To define ECB for sequenicng problems, consider a reference economy where there are no transfers, agents arrive randomly, each arrival order is equally likely, and the facility starts processing jobs once all the jobs arrive. ECB requires that the utility of each agent is no less than the expected cost of the agent associated with random arrival where each arrival order is equally likely. For queueing problems, the notions of ICB and

[5]For the queueing problem this impossibility was shown by Chun, Mitra and Mutuswami [7] and our result generalizes it to the sequencing problems.

[6]See Bevia [1], Moulin [32], [33], Steinhaus [36] and Yengin [39].

ECB are equivalent. For all sequencing problems with ICB and ECB, both the constrained acceptability property as well as the weighted net acceptability property get satisfied. Given these two properties, we obtain the relative pivotal mechanisms with ICB and ECB. We also show that for both these bounds, when there are three agents, we can get feasible relative pivotal mechanisms only for queueing problems.

1.5. **Implication in terms of queueing problems.** For the queueing problems with acceptable utility bounds satisfying the constrained acceptability property, one can give a more explicit form of the transfers associated with the relative pivotal mechanism. We first characterize the set of all mechanisms satisfying outcome efficiency, strategyproofness and ICB (ECB). For more than two agents, we also characterize the set of all mechanisms satisfying outcome efficiency, strategyproofness, ICB (ECB) and budget balance. Using this result, we provide a sufficient restriction on the acceptability parameter that guarantees the existence of balanced relative pivotal mechanisms. The sufficiency condition also becomes necessary when acceptability parameters are equal across agents.

2. The framework

Consider a finite set of agents $N = \{1, 2, \ldots, n\}$ who want to process their jobs using a facility that can be used sequentially. The job processing time can be different for different agents. Specifically, for each agent $i \in N$, the job processing time is given by $s_i > 0$. Let $\theta_i S_i$ measure the cost of job completion for agent $i \in N$ where $S_i \in \mathbb{R}_{++}$ is the job completion time for this agent and $\theta_i \in \Theta := \mathbb{R}_{++}$ denotes his constant per-period waiting cost where $\mathbb{R}_{++}$ is the positive orthant of the real line $\mathbb{R}$. Due to the sequential nature of providing the service, the job completion time for agent i depends not only on his own processing time s_i, but also on the processing time of the agents who precede him in the order of service. By means of an order $\sigma = (\sigma_1, \ldots, \sigma_n)$ on N, one can describe the position of each agent in the order. Specifically, $\sigma_i = k$ indicates that agent i has the k-th position in the order. Let Σ be the set of $n!$ possible orders on N. We define $P_i(\sigma) = \{j \in N \setminus \{i\} \mid \sigma_j < \sigma_i\}$ to be the predecessor set of i in the order

σ. Similarly, $F_i(\sigma) = \{j \in N \setminus \{i\} \mid \sigma_j > \sigma_i\}$ denotes the follower (or successor) set of i in the order σ. Given a vector $s = (s_1, \ldots, s_n) \in \mathbb{R}^n_{++}$ and an order $\sigma \in \Sigma$, the cost of job completion for agent $i \in N$ is $\theta_i S_i(\sigma)$, where the job completion time is $S_i(\sigma) = \sum_{j \in P_i(\sigma)} s_j + s_i$. Note that, for any $i \in N$ we write, $\sum_{j \in P_i(\sigma)} s_j = 0$ if $P_i(\sigma) = \emptyset$. The agents have quasi-linear utility of the form $u_i(\sigma, \tau_i; \theta_i) = -\theta_i S_i(\sigma) + \tau_i$ where σ is the order, $\tau_i \in \mathbb{R}$ is the transfer that he receives and the parameter of the model θ_i is the waiting cost. Given any processing time vector $s = (s_1, \ldots, s_n) \in \mathbb{R}^n_{++}$ define $A(s) = \sum_{j \in N} s_j$ and, with slight abuse of notation, we denote a *sequencing problem* by Ω and we denote the set of all sequencing problems with the set of agents N by $\mathcal{S}(N)$. A sequencing problem $\Omega \in \mathcal{S}(N)$ is called a *queueing problem* if $s = (s_1, \ldots, s_n)$ is such that $s_1 = \ldots = s_n$. We denotes the set of all queueing problems with the set of agents N by $\mathcal{Q}(N)$. Clearly, $Q(N) \subset \mathcal{S}(N)$ for any given N (such that N is a finite set and $n \geq 2$).

A typical profile of waiting costs is denoted by $\theta = (\theta_1, \ldots, \theta_n) \in \Theta^n$. For any $i \in N$, let θ_{-i}, denote the profile $(\theta_1 \ldots \theta_{i-1}, \theta_{i+1}, \ldots \theta_n) \in \Theta^{n-1}$ which is obtained from the profile θ by eliminating i's waiting cost. A mechanism $\mu = (\sigma, \tau)$ constitutes of a sequencing rule σ and a transfer rule τ. A *sequencing rule* is a function $\sigma : \Theta^n \to \Sigma$ that specifies for each profile $\theta \in \Theta^n$ a unique order $\sigma(\theta) = (\sigma_1(\theta), \ldots, \sigma_n(\theta)) \in \Sigma$. Because the sequencing rule is a function (and not a correspondence) we will require a tie-breaking rule to reduce a correspondence to a function which, unless explicitly discussed, is assumed to be fixed. We use the following tie-breaking rule. We take the linear order $1 \succ 2 \succ \ldots \succ n$ on the set of agents N. For any sequencing rule σ and any profile $\theta \in \Theta^n$ with a tie situation between agents $i, j \in N$, we pick the order $\sigma(\theta)$ with $\sigma_i(\theta) < \sigma_j(\theta)$ if and only if $i \succ j$. A *transfer rule* is a function $\tau : \Theta^n \to \mathbb{R}^n$ that specifies for each profile $\theta \in \Theta^n$ a transfer vector $\tau(\theta) = (\tau_1(\theta), \ldots, \tau_n(\theta)) \in \mathbb{R}^n$. Specifically, given any mechanism $\mu = (\sigma, \tau)$, if (θ'_i, θ_{-i}) is the announced profile when the true waiting cost of i is θ_i, then utility of i is $u_i(\mu_i(\theta'_i, \theta_{-i}); \theta_i) = -\theta_i S_i(\sigma(\theta'_i, \theta_{-i})) + \tau_i(\theta'_i, \theta_{-i})$ where $\mu_i(\theta'_i, \theta_{-i}) := (\sigma(\theta'_i, \theta_{-i}), \tau_i(\theta'_i, \theta_{-i}))$. Given any $\Omega \in \mathcal{S}(N)$, any $\theta \in \Theta^n$ and any order $\sigma \in \Sigma$, define the aggregate cost as $C(\sigma; \theta)$, that is, $C(\sigma; \theta) := \sum_{j \in N} \theta_j S_j(\sigma)$.

Definition 1. A sequencing rule σ^* is *outcome efficient* if for any $\theta \in \Theta^n$, $\sigma^*(\theta) \in \text{argmin}_{\sigma \in \Sigma} C(\sigma; \theta)$.

The ratio of the waiting cost and processing time of any agent i, that is, θ_i / s_i is known as the urgency index. From Smith [35] it follows that σ^* is outcome efficient if and only if the following holds: **(OE)** For any $\theta \in \Theta^n$, the selected order $\sigma^*(\theta)$ satisfies the following: For any $i, j \in N$, $\theta_i / s_i > \theta_j / s_j \Rightarrow \sigma^*_i(\theta) < \sigma^*_j(\theta)$. We say that a mechanism $\mu = (\sigma, \tau)$ satisfies outcome efficiency if $\sigma = \sigma^*$.

Suppose that a waiting cost of zero was admissible in the domain. Consider any outcome efficient order $\sigma^*(\theta)$ for $\theta \in \Theta^n$. We define the "induced" order $\sigma^*(0, \theta_{-i})$ as follows:

$$\sigma^*_j(0, \theta_{-i}) = \begin{cases} \sigma^*_j(\theta) - 1 & \text{if } j \in F_i(\sigma^*(\theta)), \\ \sigma^*_j(\theta) & \text{if } j \in P_i(\sigma^*(\theta)), \\ n & j = i \end{cases} \tag{1}$$

In words, given $\theta \in \Theta^n$ and given any $i \in N$, $\sigma^*(0, \theta_{-i})$ is the order formed by setting the waiting cost of agent i at zero and hence moving agent i to the last position (following the outcome efficiency condition of Smith [35] by admitting zero waiting cost of agent i) so that only the agents in the set behind $F_i(\sigma^*(\theta))$ move up by one position under the outcome efficient queue for the induced profile $(0, \theta_{-i})$.

Definition 2. For a sequencing rule σ, a mechanism $\mu = (\sigma, \tau)$ is *strategyproof* (dominant strategy incentive compatible) if the transfer rule $\tau : \Theta^n \to \mathbb{R}^n$ is such that for any $i \in N$, any $\theta_i, \theta'_i \in \Theta$ and any $\theta_{-i} \in \Theta^{n-1}$,

$$u_i(\mu_i(\theta); \theta_i) \geq u_i(\mu_i(\theta'_i, \theta_{-i}); \theta_i). \tag{2}$$

For a given sequencing rule σ, strategyproofness of a mechanism $\mu = (\sigma, \tau)$ requires that the transfer rule τ is such that truthful reporting for any agent weakly dominates false reporting no matter what others' report.

Definition 3. A mechanism μ satisfies *feasibility* if for any $\theta \in \Theta^n$, $\sum_{j \in N} \tau_i(\theta) \leq 0$.

Definition 4. A mechanism μ satisfies *budget balance* if for any $\theta \in \Theta^n$, $\sum_{j \in N} \tau_i(\theta) = 0$.

2.1. **Acceptable utility bounds.** Given any sequencing problem $\Omega \in \mathcal{S}(N)$, let $O_i(s)$ be the acceptability parameter of agent i. Let $O(N;s) := (O_1(s), \dots, O_n(s)) \in \mathbb{R}^n$ denote the acceptability parameter vector. We represent a typical sequencing problem with acceptable utility bounds by $\Gamma = (\Omega, O(N;s))$ where $\Omega \in \mathcal{S}(N)$ and the associated $O(N;s) \in \mathbb{R}^n$ is the acceptable utility bounds vector.

Definition 5. For Γ, a mechanism $\mu = (\sigma, \tau)$ satisfies *acceptable utility bounds* if the transfer rule $\tau : \Theta^n \to \mathbb{R}^n$ is such that for any $i \in N$, any $\theta_i \in \Theta$ and any $\theta_{-i} \in \Theta^{n-1}$,

$$u_i(\mu_i(\theta_i, \theta_{-i}); \theta_i) \geq -\theta_i O_i(s). \tag{3}$$

3. Acceptable utility bounds, outcome efficiency and strategyproofness

Given any sequencing problem with acceptable utility bounds $\Gamma = (\Omega, O(N;s))$, we first try to identify the restriction on $O(N;s)$ for which we can get a mechanism satisfying outcome efficiency, strategyproofness and acceptable utility bounds. The property defined below puts a constraint on the acceptability parameter, indicating that an agent will always need to incur atleast the cost of his own processing time. Thus, the acceptable utility bound is no less than the cost of serving that agent when he occupies the first position in the queue.

Definition 6. Any sequencing problem with acceptable utility bounds $\Gamma = (\Omega, O(N;s))$ satisfies the *constrained acceptability property* if $O(N;s) = (O_1(s), \dots, O_n(s))$ is such that

$$O_i(s) \geq s_i \;\; \forall\, i \in N. \tag{4}$$

Let $\mathcal{G}(N)$ be the set of all Γ satisfying the constrained acceptability property given by condition (4).

Theorem 1. The following statements are equivalent:

(SPC1) For a Γ we can find a mechanism that satisfies outcome efficiency, strategyproofness and acceptable utility bounds.

(SPC2) Γ satisfies the constrained acceptability property, that is, $\Gamma \in \mathcal{G}(N)$.

Given any $\Gamma \in \mathcal{G}(N)$ what is the set of all mechanisms that satisfy outcome efficiency, strategyproofness and acceptable utility bounds? The next result answers this question. Before going to the result we introduce some notations and definitions. For any agent $i \in N$ and any given profile $\theta_{-i} \in \Theta^{n-1}$, define the function

$$T_i(x_i;\theta_{-i}) := \sum_{j \in N \setminus \{i\}} \theta_j S_j(\sigma^*(x_i,\theta_{-i})) + \{S_i(\sigma^*(x_i,\theta_{-i})) - O_i(s)\}x_i, \tag{5}$$

for $x_i \in \mathbb{R}_+$. Observe that if $O_i(s) > A(s) = \sum_{j \in N} s_j$, then $S_i(\sigma^*(x_i,\theta_{-i})) < O_i(s)$ for all $x_i \in \Theta$ and hence the function $T_i(x_i;\theta_{-i})$ has no maximum value $x_i \in \Theta$ though the function has a least upper bound if we set $x_i = 0$. Hence, if $O_i(s) > A(s)$, we have $T_i(x_i;\theta_{-i}) < T_i(0;\theta_{-i}) < \infty$ for all $x_i \in \Theta$.[7] One can also verify that even if $O_i(s) = A(s)$, we have $T_i(x_i;\theta_{-i}) \leq T_i(0;\theta_{-i}) < \infty$ for all $x_i \in \Theta$. However, if $O_i(s) < s_i$, then $S_i(\sigma^*(x_i,\theta_{-i})) > O_i(s)$ for all $x_i \in \Theta$ and the function $T_i(x_i;\theta_{-i})$ has neither a maximum nor a least upper bound. Hence, for the function $T_i(x_i;\theta_{-i})$ defined on $x_i \in \Theta$ to have a least upper bound, the constrained acceptability property (of Definition 6) is necessary.

Definition 7. An outcome efficient mechanism $\mu^p = (\sigma^*, \tau^p)$ is called a *relative pivotal mechanism* if τ^p satisfies the following property: For any profile $\theta \in \Theta^n$ and any agent $i \in N$,

$$\tau_i^p(\theta) = \{S_i(\sigma^*(\theta_i^*,\theta_{-i})) - O_i(s)\}\theta_i^* + RP_i(\theta) + h_i(\theta_{-i}), \tag{6}$$

where, given the function $T_i(x_i;\theta_{-i})$ (defined in (5)), $\theta_i^* \in \mathbb{R}_+$ is such that $T_i(\theta_i^*;\theta_{-i}) \geq T_i(x_i;\theta_{-i})$ for all $x_i \in \Theta$, $RP_i(\theta) := \sum_{j \in N \setminus \{i\}} (|P_j(\sigma^*(\theta_i^*,\theta_{-i}))| - |P_j(\sigma^*(\theta))|)\theta_j s_i$ and $h_i : \Theta^{|N \setminus \{i\}|} \to \mathbb{R}_+$.

Let $\mathcal{R}(N)$ denote the set of all relative pivotal mechanisms defined in Definition 7.

Theorem 2. For any $\Gamma \in \mathcal{G}(N)$, an outcome efficient mechanism $\mu = (\sigma^*, \tau)$ satisfies strategyproofness and acceptable utility bounds if and only if it is a relative pivotal mechanism, that is, $\mu \in \mathcal{R}(N)$.

[7]Given (1), the order $\sigma^*(0;\theta_{-i})$ is well-defined and hence the function $T_i(x_i;\theta_{-i})$ is well-defined at $x_i = 0$.

We try and explain Definition 7 and Theorem 2. It is well-known from Holmström [24] that for outcome efficiency and strategyproof it is necessary that the mechanism $\mu = (\sigma^*, \tau)$ must be a VCG mechanism where the transfers satisfy the following property: For any profile $\theta \in \Theta^n$ and any agent $i \in N$, $\tau_i(\theta) = -C(\sigma^*(\theta);\theta) + \theta_i S_i(\sigma^*(\theta)) + g_i(\theta_{-i})$ where $g_i : \Theta^{|N \setminus \{i\}|} \to \mathbb{R}$ is arbitrary. The relative pivotal mechanism given in Definition 7 is a VCG mechanism which is obtained for each agent $i \in N$ and each profile $\theta \in \Theta^n$ by substituting $g_i(\theta_{-i}) = T_i(\theta_i^*;\theta_{-i}) + h_i(\theta_{-i})$ where $T_i(\theta_i^*;\theta_{-i})$ (resulting from the optimization exercise in Definition 7) and the restriction $h_i(\theta_{-i}) \geq 0$ are necessary to satisfy the acceptable utility bounds. After appropriate simplification of the VCG transfer $\tau_i(\theta) = -C(\sigma^*(\theta);\theta) + \theta_i S_i(\sigma^*(\theta)) + g_i(\theta_{-i})$ by using $g_i(\theta_{-i}) = T_i(\theta_i^*;\theta_{-i}) + h_i(\theta_{-i})$ we get that for all $\theta \in \Theta^n$ and all $i \in N$,

$$\tau_i^p(\theta) = -C(\sigma^*(\theta);\theta) + \theta_i S_i(\sigma^*(\theta)) + T_i(\theta_i^*;\theta_{-i}) + h_i(\theta_{-i}). \tag{7}$$

Simplifying (7) we get a subset of VCG mechanisms which we call relative pivotal mechanisms (Definition 7). From the proof of Theorem 2 it is clear that given any relative pivotal mechanism $\mu^p = (\sigma^*, \tau^p) \in \mathcal{R}(N)$, for any $\theta \in \Theta^n$ and any $i \in N$, $u_i(\mu_i^p(\theta_i, \theta_{-i});\theta_i) = -\theta_i O_i(s) + \{T_i(\theta_i^*;\theta_{-i}) - T_i(\theta_i;\theta_{-i}) + h_i(\theta_{-i})\} \geq -\theta_i O_i(s)$ since $T_i(\theta_i^*;\theta_{-i}) - T_i(\theta_i;\theta_{-i}) + h_i(\theta_{-i}) \geq 0$. Hence, acceptable utility bounds is satisfied for all agents.

The sum $RP_i(\theta) = \sum_{j \in N \setminus \{i\}} (|P_j(\sigma^*(\theta_i^*, \theta_{-i}))| - |P_j(\sigma^*(\theta))|)\theta_j s_i$ in condition (6) captures the relative pivotal nature of this sub-class of VCG mechanisms. Given any profile $i \in N$, any $\theta_{-i} \in \Theta^{n-1}$ the 'benchmark' type θ_i^* of agent i is obtained from the optimization exercise in Definition 7 and if this θ_i^* is taken along with $\theta_{-i} \in \Theta^{n-1}$, then the resulting benchmark outcome efficient order is $\sigma^*(\theta_i^*, \theta_{-i})$. Given any $\theta_i \in \Theta$, this benchmark order $\sigma^*(\theta_i^*, \theta_{-i})$ may or may not be the same as the actual outcome efficient order $\sigma^*(\theta_i, \theta_{-i})$ though the relative order across the agents other than i remains unchanged.[8] Given $\sigma^*(\theta_i^*, \theta_{-i})$ and $\sigma^*(\theta_i, \theta_{-i})$, we can have the three mutually exclusive and exhaustive possibilities-(i) $P_i(\sigma^*(\theta_i, \theta_{-i})) \subset$

[8]Specifically, for any $\sigma^*(\theta_i^*, \theta_{-i})$ and $\sigma^*(\theta_i, \theta_{-i})$, the relative order across the agents other than i remains unchanged means that for any $j, k \in N \setminus \{i\}$ with $j \neq k$, $\sigma_j^*(\theta_i^*, \theta_{-i}) > \sigma_k^*(\theta_i^*, \theta_{-i})$ if and only if and $\sigma_j^*(\theta_i, \theta_{-i}) > \sigma_k^*(\theta_i, \theta_{-i})$.

$P_i(\sigma^*(\theta_i^*,\theta_{-i}))$, (ii) $P_i(\sigma^*(\theta_i,\theta_{-i})) = P_i(\sigma^*(\theta_i^*,\theta_{-i}))$, and, (iii) $P_i(\sigma^*(\theta_i^*,\theta_{-i})) \subset P_i(\sigma^*(\theta_i,\theta_{-i}))$.

(R1) If $P_i(\sigma^*(\theta_i,\theta_{-i})) \subset P_i(\sigma^*(\theta_i^*,\theta_{-i}))$ (so that $\theta_i^* \in [0,\theta_i)$), then relative to $\sigma^*(\theta_i^*,\theta_{-i})$, agent i has inflicted an incremental cost of $\theta_j s_i$ to each agent $j \in P_i(\sigma^*(\theta_i^*,\theta_{-i}) \setminus P_i(\sigma^*(\theta_i,\theta_{-i}))$ under the actual order $\sigma^*(\theta_i,\theta_{-i})$. Hence, for any $j \in P_i(\sigma^*(\theta_i^*,\theta_{-i}) \setminus P_i(\sigma^*(\theta_i,\theta_{-i}))$, we get $|P_j(\sigma^*(\theta_i^*,\theta_{-i}))| - |P_j(\sigma^*(\theta_i,\theta_{-i}))| = -1$. Therefore, using the sum in (6) it follows that agent i has to pay

$$RP_i(\theta) = \sum_{j\in N\setminus\{i\}} (|P_j(\sigma^*(\theta_i^*,\theta_{-i}))| - |P_j(\sigma^*(\theta_i,\theta_{-i}))|)\theta_j s_i = -\sum_{j\in P_i(\sigma^*(\theta_i^*,\theta_{-i})\setminus P_i(\sigma^*(\theta_i,\theta_{-i}))} \theta_j s_i.$$

When can we have $\theta_i^* = 0$? If for any agent $i \in N$ we have $O_i(s) \geq A(s)$, then for every $\theta_{-i} \in \Theta^{n-1}$, $T_i(x_i;\theta_{-i})$ is decreasing in $x_i \in \Theta$ implying that by setting $\theta_i^* = 0$ we get $T_i(0;\theta_{-i}) \geq T_i(x_i,\theta_{-i})$ for all $x_i \in \Theta$. In this case,

$$RP_i(\theta) = \sum_{j\in N\setminus\{i\}} (|P_j(\sigma^*(0,\theta_{-i}))| - |P_j(\sigma^*(\theta_i,\theta_{-i}))|)\theta_j s_i = -\sum_{j\in F_i(\sigma^*(\theta_i^*,\theta_{-i})} \theta_j s_i.$$

(R2) If $P_i(\sigma^*(\theta_i,\theta_{-i})) = P_i(\sigma^*(\theta_i^*,\theta_{-i}))$, then $\sigma^*(\theta_i^*,\theta_{-i}) = \sigma^*(\theta_i,\theta_{-i})$ and agent i has neither inflicted any incremental cost to any other agent nor has agent i induced any incremental benefit for any other agent, that is, $|P_j(\sigma^*(\theta_i^*,\theta_{-i}))| = |P_j(\sigma^*(\theta_i,\theta_{-i}))|$ for all $j \in N$. Hence, using the sum in (6), it follows that

$$RP_i(\theta) = \sum_{j\in N\setminus\{i\}} (|P_j(\sigma^*(\theta_i,\theta_{-i}))| - |P_j(\sigma^*(\theta_i^*,\theta_{-i}))|)\theta_j s_i = 0$$

.

(R3) If $P_i(\sigma^*(\theta_i^*,\theta_{-i})) \subset P_i(\sigma^*(\theta_i,\theta_{-i}))$ (so that $\theta_i^* > \theta_i$), then relative to the outcome efficient order $\sigma^*(\theta_i^*,\theta_{-i})$, agent i has given an incremental benefit of $\theta_j s_i$ to each $j \in P_i(\sigma^*(\theta_i,\theta_{-i})) \setminus P_i(\sigma^*(\theta_i^*,\theta_{-i})$ under the outcome efficient order $\sigma^*(\theta_i,\theta_{-i})$. Hence, for any $j \in P_i(\sigma^*(\theta_i,\theta_{-i})) \setminus P_i(\sigma^*(\theta_i^*,\theta_{-i})$, we have $|P_j(\sigma^*(\theta_i^*,\theta_{-i}))| - |P_j(\sigma^*(\theta_i,\theta_{-i}))| = 1$. Thus, from the sum in (6), it follows that agent i gets a reward of

$$RP_i(\theta) = \sum_{j\in N\setminus\{i\}} (|P_j(\sigma^*(\theta_i^*,\theta_{-i}))| - |P_j(\sigma^*(\theta_i,\theta_{-i}))|)\theta_j s_i = \sum_{j\in P_i(\sigma^*(\theta_i,\theta_{-i}))\setminus P_i(\sigma^*(\theta_i^*,\theta_{-i})} \theta_j s_i.$$

Therefore, (R1), (R2) and (R3) explains how the sum $RP_i(\theta)$ in (6) for agent i with type θ_i, given θ_{-i} is calculated based on the difference in the cost of all other agents $N \setminus \{i\}$ that results from the actual profile specific outcome efficient order $\sigma^*(\theta_i, \theta_{-i})$ relative to the benchmark outcome efficient order $\sigma^*(\theta_i^*, \theta_{-i})$. What follows from the above discussion is that for all $\theta \in \Theta^n$ and each $i \in N$, either $|P_j(\sigma^*(\theta_i^*, \theta_{-i}))| - |P_j(\sigma^*(\theta))| \in \{-1, 0\}$ for all $j \in N \setminus \{i\}$ or $|P_j(\sigma^*(\theta_i^*, \theta_{-i}))| - |P_j(\sigma^*(\theta))| \in \{0, 1\}$ for all $j \in N \setminus \{i\}$. Equivalently, we cannot find a profile $\theta \in \Theta^n$ and an agent $i \in N$ such that $|P_j(\sigma^*(\theta_i^*, \theta_{-i}))| - |P_j(\sigma^*(\theta))| = -1$ for some agent $j \in N \setminus \{i\}$ and $|P_k(\sigma^*(\theta_i^*, \theta_{-i}))| - |P_k(\sigma^*(\theta))| = 1$ for other agent $k \in N \setminus \{i, j\}$.

3.1. **Feasibility and budget balance.** Before going to our results on identifying relative pivotal mechanisms that ensures outcome efficiency, strategyproofness, acceptable utility bounds and feasibility, we first drop the strategyproofness requirement and provide a necessary restriction for getting mechanisms that satisfy outcome efficiency, acceptable utility bounds and feasibility.

Definition 8. A sequencing problem with acceptable utility bounds $\Gamma = (\Omega, O(N, s)) \in \mathcal{G}(N)$ satisfies the property of *weighted net acceptability* if

$$\mathcal{D}(O(N,s)) := \sum_{j \in N} s_j \left\{ O_j(s) - \left(\frac{s_j + A(s)}{2} \right) \right\} \geq 0. \tag{8}$$

For any sequencing problem with acceptable utility bounds $\Gamma = (\Omega, O(N, s))$ with $O_i(s) = s_i$ for all $i \in N$, condition (8) fails to hold. For any sequencing problem with acceptable utility bounds $\Gamma = (\Omega, O(N, s))$ with $O_i(s) \geq (s_i + A(s))/2$ for all $i \in N$, condition (8) is satisfied. Let $\overline{\mathcal{G}}(N) (\subset \mathcal{G}(N))$ denote the set of all sequencing problems with acceptable utility bounds satisfying the constrained acceptability property and the weighted net acceptability.

Lemma 1. If for any $\Gamma = (\Omega, O(N, s)) \in \mathcal{G}(N)$, we can find a mechanism that satisfies outcome efficiency, acceptable utility bounds and feasibility, then Γ must satisfy the weighted net acceptability, that is, $\Gamma \in \overline{\mathcal{G}}(N)$.

Remark 1. For any sequencing problem with acceptable utility bounds $\Gamma = (\Omega, O(N, s))$, a good way to explain condition (8) is in terms of mean $\mu(s)$, variance $V(s)$ and coefficient of variation $CoV(s) := \sqrt{V(s)}/\mu$ of the elements of the processing time vector $s = (s_1, \ldots, s_n)$. Specifically, an equivalent way of representing condition (8) is the following:

$$\sum_{j \in N} w_j(s) O_j(s) \geq \frac{\mu(s)}{2} \left[n + 1 + \{CoV(s)\}^2 \right], \tag{9}$$

where $w_i(s) := s_i / A(s)$ for all $i \in N$.[9]

(i) If we have the queueing problem, that is if $\Omega \in \mathcal{Q}(N)$ with $s_1 = \ldots = s_n = a > 0$, then $\mu(s) = a$, $CoV(s) = 0$ and $w_i(s) = 1/n$ for all $i \in N$. Condition (9) holds if and only if $\sum_{j \in N} O_j(s)/n \geq (n+1)a/2$. Moreover, if we also require that the acceptable utility bound of all the agents are identical, that is $O_i(s) = B^*$ for all $i \in N$, then condition (9) requires $B^* \geq (n+1)a/2$.

(ii) It is well-known that $CoV(s) \leq \sqrt{n-1}$ for any positive integer n and any $s = (s_1, \ldots, s_n) \in \mathbb{R}^n_{++}$. Therefore, a sufficient condition for (9) to hold for any sequencing problem with acceptable utility bounds $\Gamma = (\Omega, O(N, s))$ is obtained by substituting $CoV(s) = \sqrt{n-1}$ in (9) that yields $\sum_{j \in N} w_j(s) O_j(s) \geq n\mu(s) = A(s)$.

Remark 2. Fix any N and any $s = (s_1, \ldots, s_n) \in \mathbb{R}^n_{++}$. Let $\mathcal{O}(N, s)$ denote the set of acceptable utility bounds vectors $O(N, s) = (O_1(s), \ldots, O_n(s))$ satisfying the constrained acceptability property and the weighted net acceptability. It is obvious that the set $\mathcal{O}(N, s)$ is non-empty and convex. It is non-empty since for $\bar{O}(N, s) = (\bar{O}_1(s), \ldots, \bar{O}_n(s))$ with $\bar{O}_i(s) = (s_i + A(s))/2$ for all $i \in N$, inequality (8) holds. For convexity of $\mathcal{O}(N, s)$, observe that if $O(N, s), O'(N, s) \in \mathcal{O}(N, s)$ so that $\mathcal{D}(O(N, s)) \geq 0$ and $\mathcal{D}(O'(N, s)) \geq 0$, then, given (8) it easily follows that for any $\lambda^* \in [0, 1]$ we get $\mathcal{D}(\lambda^* O(N, s) + (1 - \lambda^*) O'(N, s)) = \lambda^* \mathcal{D}(O(N, s)) + (1 - \lambda^*) \mathcal{D}(O'(N, s)) \geq 0$ implying $\lambda^* O(N, s) + (1 - \lambda^*) O'(N, s) \in \mathcal{O}(N, s)$. For any $i \in N$, define $E_i(s) := s_i + \left(\sum_{j \in N} s_j \sum_{k \in N \setminus \{j\}} s_k \right) / s_i$ and $O^i(N, s) := (E_i(s), s_{-i})$.[10] It is easy to verify that for any $i \in N$, $O^i(N, s) = (E_i(s), s_{-i}) \in \mathcal{O}(N, s)$ since $\mathcal{D}(O^i(N, s)) = 0$. Moreover, given (8) it is also obvious that for any $i \in N$ and any $\underline{O}(N, s) \in \mathcal{R}^n_{++}$ such that $O^i(N, s) \geq \underline{O}(N, s)$ and $\underline{O}(N, s) \neq O^i(N, s)$, we have $\underline{O}(N, s) \notin \mathcal{O}(N, s)$. Therefore, for any $i \in N$, $O^i(N, s)$ is a boundary point of the set

[9] To derive inequality (9) we have used the following equalities: $\sum_{j \in N} s_j^2 = nVar(s) + n\{\mu(s)\}^2 = n\{\mu(s)\}^2\{1 + Cov(s)\} = A(s)\mu(s)\{1 + Cov(s)\}$.

[10] Note that if $|N| = 2$, then $E_i = A(s)$ for any $i \in N$.

$\mathcal{O}(N,s)$. Further, for the same type of reasoning, $\bar{O}(N,s) = (\bar{O}_1(s),\ldots,\bar{O}_n(s)) \in \mathcal{O}(N,s)$ such that $\bar{O}_i(s) = (s_i + A(s))/2$ for all $i \in N$ is also a boundary point of $\mathcal{O}(N,s)$. However, one can verify that $\sum_{j\in N} w_j(s)O^i(N,s) = \bar{O}(N,s)$, that is, $\bar{O}(N,s)$ is a weighted sum of the elements of the set $\{\{O^i(N,s)\}_{i\in N}\}$ with weight $w_i(s) = s_i/A(s)$ for each $i \in N$. The set $\{\{O^i(N,s)\}_{i\in N}\}$ plays a key role in explaining the set $\mathcal{O}(N,s)$. For any $\lambda = (\lambda_1,\ldots,\lambda_n) \in [0,1]^n$ with $\sum_{j\in N}\lambda_j = 1$, consider the vector $\sum_{j\in N}\lambda_j O^j(N,s) = (\lambda_1 E_1(s) + (1-\lambda_1)s_1,\ldots,\lambda_n E_n(s) + (1-\lambda_n)s_n)$. One can verify that $\mathcal{O}(N,s)$ is a non-empty and convex set given by

$$\mathcal{O}(N,s) = \left\{ O(N,s) \in \mathbb{R}^N_{++} \mid \exists \lambda \in [0,1]^n \text{ with } \sum_{j\in N}\lambda_j = 1, \text{ s.t. } O(N,s) \geq \sum_{j\in N}\lambda_j O^j(N,s) \right\}. \tag{10}$$

Therefore, the set $\mathcal{O}(N,s)$ is non-empty and convex with the added property that any element in this set weakly vector dominates some weighted sum of the elements of the set $\{\{O^i(N,s)\}_{i\in N}\}$.

Given Lemma 1, from now on we restrict our attention only to the set $\overline{\mathcal{G}}(N)$ of all sequencing problems with acceptable utility bounds satisfying the constrained acceptability property and the weighted net acceptability, that is, we consider any $\Gamma = (\Omega, O(N,s))$ such that $O(N,s) \in \mathcal{O}(N,s)$ and the set $\mathcal{O}(N,s)$ is given by (10) of Remark 2.

Definition 9. An outcome efficient mechanism $\hat{\mu}^p = (\sigma^*, \hat{\tau}^p)$ is called a *minimal relative pivotal mechanism* if it is a relative pivotal mechanism with the property that for all $i \in N$ and all $\theta_{-i} \in \Theta^{n-1}$, $h_i(\theta_{-i}) = 0$, that is, for any profile $\theta \in \Theta^n$ and any agent $i \in N$,

$$\hat{\tau}^p_i(\theta) = \{S_i(\sigma^*(\theta^*_i, \theta_{-i})) - O_i(s)\}\theta^*_i + RP_i(\theta), \tag{11}$$

where $\theta^*_i \in \mathbb{R}_+$ ensures $T_i(\theta^*_i;\theta_{-i}) \geq T_i(x_i;\theta_{-i})$ for all $x_i \in \Theta$ and $RP_i(\theta) = \sum_{j\in N\setminus\{i\}}(|P_j(\sigma^*(\theta^*_i,\theta_{-i}))| - |P_j(\sigma^*(\theta))|)\theta_j s_i$.

Observe that if a relative pivotal mechanism $\mu^p = (\sigma^*, \tau^p) \in \mathcal{R}(N)$ is feasible, then the minimal relative pivotal mechanism $\hat{\mu}^p = (\sigma^*, \hat{\tau}^p)$ is also feasible since for any $\theta \in \Theta^n$ and any $i \in N$, $\tau^p_i(\theta) - \hat{\tau}^p_i(\theta) = h_i(\theta_{-i}) \geq 0$. Therefore, for any $\Gamma \in \mathcal{G}(N)$, if we want to check whether we can find a feasible relative pivotal mechanism or not, we simply need to check the prospect of feasibility with the minimal relative pivotal mechanism $\hat{\mu}^p$.

Proposition 1. For any $\Gamma = (\Omega, O(N,s)) \in \overline{\mathcal{G}}(N)$ such that $|N| = 2$ we have the following results:

(B2a) A feasible relative pivotal mechanism exists if and only if $O_1(s) \geq A(s)$ and $O_2(s) \geq A(s)$.

(B2b) There is no budget balanced relative pivotal mechanism.

Can we find budget balanced relative pivotal mechanisms for sequencing problems with acceptable utility bounds satisfying the constrained acceptability and the weighted net acceptability when there are more than two agents?

Proposition 2. For any $\Gamma = (\Omega, O(N,s)) \in \overline{\mathcal{G}}(N)$ such that $|N| \geq 3$ and $O_i(s) \geq A(s)$ for all $i \in N$, we can find budget balanced relative pivotal mechanisms.

Remark 1 (ii) states that the *weighted average of the acceptable utility bounds is no less than the aggregate processing time* (specifically, $\sum_{j \in N} w_j(s) O_j(s) \geq A(s)$) is a sufficient condition for weighted net acceptability property. Proposition 1 shows that the *actual acceptable utility bound of each agent is no less than the aggregate processing time* is necessary and sufficient for feasibility relative pivotal mechanisms when there are two agents and Proposition 2 shows that the same condition is sufficient to get budget balanced relative pivotal mechanism when there are more than two agents. What can we say about obtaining feasible relative pivotal mechanism for any $\Gamma = (\Omega, O(N,s)) \in \overline{\mathcal{G}}(N)$ such that $|N| \geq 3$ and there exists at least one agent with $O_i(s) \in (s_i, A(s))$? It is difficult to answer this question in general as the transfers associated with any relative pivotal mechanism lacks closed form representation. However, the following example suggests that one would expect to get more restriction on the processing time of the agents (over and above what is required under the constrained acceptability and weighted net acceptability properties) to get feasible relative pivotal mechanisms.

Example 1. Consider any $\Gamma = (\Omega, O(N,s)) \in \mathcal{G}(N)$ such that $|N| = 3$ and $O_i(s) = s_i + \max_{j \neq i} s_j$ for all $i \in N$. Without loss of generality, assume that $s_1 \geq s_2 \geq s_3$. Observe that condition (8) holds since $\mathcal{D}(s) = s_1(s_2 - s_3)/2 + s_2(s_1 - s_3)/2 +$

$s_3(s_1 - s_2)/2 \geq 0$. Hence, $\Gamma = (\Omega, O(N,s)) \in \overline{\mathcal{G}}(N)$. Consider the profile $\theta \in \Theta^3$ such that $\sigma_j^*(\theta) = n + 1 - j$ for all $j \in N$ and in particular $\theta_3/s_3 = a > \theta_2/s_2 = b > \theta_1/s_1 = c > 0$. Using the function $T_i(x_i; \theta_{-i})$ (in (5)), we can fix $\theta_1^* = s_1 b$, $\theta_2^* = s_2 c$ and $\theta_3^* = s_3 c$. Then, using the transfers associated with the minimal relative pivotal mechanism (Definition 9), we get the following:

(1) $\hat{\tau}_1(\theta) = s_1 s_3 b$,
(2) $\hat{\tau}_2(\theta) = -cs_2(s_1 - s_3)$, and
(3) $\hat{\tau}_3(\theta) = -cs_3(s_1 - s_2) - s_3 s_2 b$.

If $s_1 > s_2$ and $a > b > c + c[s_2(s_1 - s_3)/s_3(s_1 - s_2)]$, then $\sum_{j \in N} \hat{\tau}_j(\theta) = (b - c)s_3(s_1 - s_2) - cs_2(s_1 - s_3) > 0$ and feasibility gets violated. Hence, for feasibility to hold it is necessary that $s_1 = s_2 \geq s_3$ which is a restriction on the processing time vector $s = (s_1, s_2, s_3)$.

4. Applications

4.1. **Sequencing with a given initial order.** For a sequencing problem $\Omega \in \mathcal{S}(N)$ with initial order, there is a preexisting order in which the agents have arrived to use the facility and the job processing starts only after all agents have arrived to use the facility. This problem is the natural extension of the problem of re-ordering an existing queue (addressed by Chun, Mitra and Mutuswami [7] and by Gershkov and Schweinzer [20]) to the sequencing problem. Suppose that initial order of arrival is $\sigma^0 \in \Sigma$. In this case, the acceptable utility bounds vector is $O^{\sigma^0}(N,s) = (O_1^{\sigma^0}(s), \ldots, O_n^{\sigma^0}(s)) \in \mathbb{R}_{++}^n$ where for each $i \in N$, $O_i^{\sigma^0}(s) = s_i + \sum_{j \in P_i(\sigma^0)} s_j$ and hence for any profile $\theta \in \Theta^n$, $\sum_{j \in N} \theta_j O_j^{\sigma^0}(s) = C(\sigma^0, \theta)$. Let $\mathcal{I}(N) = \{(\Omega, O^{\sigma^0}(N,s)) \mid \Omega \in \mathcal{S}(N), \sigma^0 \in \Sigma\}$ denote the set of all sequencing problems with initial order. Every $(\Omega, O^{\sigma^0}(N,s)) \in \mathcal{I}(N)$ satisfies the constrained acceptability property since for each $i \in N$, $O_i^{\sigma^0}(s) = s_i + \sum_{j \in P_i(\sigma^0)} s_j \geq s_i$. Moreover importantly, every $(\Omega, O^{\sigma^0}(N,s)) \in \mathcal{I}(N)$ satisfies the weighted net acceptability since $\mathcal{D}(s) = \sum_{j \in N} s_j \{S_j(\sigma^0) - (s_j + A(s))/2\} = \sum_{j \in N} (s_j/2)\{(\sum_{k \in P_j(\sigma^0)} s_k - \sum_{k \in F_j(\sigma^0)} s_k\} = \sum_{j \in N} \sum_{k \in P_j(\sigma^0)} (s_j s_k/2) - \sum_{j \in N} \sum_{k \in F_j(\sigma^0)} (s_j s_k/2) = 0$ implying that condition (8) holds.[11] Hence, we get $\mathcal{I}(N) \subset \overline{\mathcal{G}}(N)$. One can check that the special feature of the relative pivotal mechanisms is that the function $T_i(x_i;\theta_{-i})$ (defined in (5)) has the following form:

$$(12) \quad T_i^I(x_i;\theta_{-i}) = \left[\sum_{j\in P_i(\sigma^*(x_i,\theta_{-i}))} s_j - \sum_{j\in P_i(\sigma^0)} s_j\right] x_i + \sum_{j\in N\setminus\{i\}} \theta_j S_j(\sigma^*(x_i,\theta_{-i})).^{12}$$

4.2. **Identical cost bounds.** Identical cost bounds (ICB) requires that each agent $i \in N$ receives at least the utility he could expect if all agents were like him (both in terms of waiting cost as well as in terms of processing time) in a reference economy. This means that each agent $i \in N$ in his reference economy has an equal chance of facing each order from Σ. Thus, ICB requires that for any agent $i \in N$ and any profile $\theta \in \Theta^n$, $u_i(\sigma(\theta),\tau_i(\theta);\theta_i) \geq -\theta_i((n+1)s_i/2)$ where $\theta_i((n+1)s_i/2)$ represents the expected cost of agent i with waiting cost θ_i and processing time s_i when all agents have the same processing time s_i and agent i gets each of the positions 1 to n with probability $1/n$. For a sequencing problem $\Omega \in \mathcal{S}(N)$ with acceptable utility bounds given by ICB, the acceptable utility bounds vector is $O^s(N,s) = (O_1^{s_1}(s),\ldots,O_n^{s_n}(s)) \in \mathbb{R}^n_{++}$ where for each $i \in N$, $O_i^{s_i}(s) = (n+1)s_i/2$. Let $\mathcal{C}(N) = \{(\Omega, O^s(N,s)) \mid \Omega \in \mathcal{S}(N)\}$ denote the set of all sequencing problems with ICB and let Γ^s represent a typical sequencing problem with ICB in $\mathcal{C}(N)$. Since for any $(\Omega, O^s(N,s)) \in \mathcal{C}(N)$, $O_i^{s_i}(s) = (n+1)s_i/2 > s_i$ for every $i \in N$, the constrained acceptability property is satisfied. Moreover, $\mathcal{D}(s) = \sum_{j\in N} s_j\{(n+1)s_j/2 - (s_j + A(s))/2\} = \sum_{j\in N} s_j\{\sum_{k\neq j}(s_j - s_k)\} = \sum_{j=1}^{n-1}\sum_{k>j}(s_j-s_k)^2 \geq 0$ and hence condition (8) also holds. Therefore, $\mathcal{C}(N) \subset \overline{\mathcal{G}}(N)$. One can easily verify that the special feature of the relative pivotal mechanisms in this context is that the function $T_i(x_i;\theta_{-i})$ (provided in (5)) has the following form:

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[11]The reason for the last equality is the following: For any two agents $j,k \in N$, $\{k \in P_j(\sigma^0) \Leftrightarrow j \in P_k(\sigma^0)\}$ which implies that for any term of the form $s_j s_k/2$, there is exactly one term of the form $-s_j s_k/2$ that cancels it out.

[12]Note that for any $i \in N$, any $\theta_{-i} \in \Theta^{n-1}$ and any $x_i \in \mathbb{R}_+$, $\{S_i(\sigma^*(x_i,\theta_{-i})) - O_i(s)\}x_i = \left[\sum_{j\in P_i(\sigma^*(x_i,\theta_{-i}))} s_j + s_i - \sum_{j\in P_i(\sigma^0)} s_j - s_i\right] x_i = \left[\sum_{j\in P_i(\sigma^*(x_i,\theta_{-i}))} s_j - \sum_{j\in P_i(\sigma^0)} s_j\right] x_i$.

$$(13)\quad T_i^C(x_i;\theta_{-i}) = \left[\sum_{j\in P_i(\sigma^*(x_i,\theta_{-i}))} s_j - \frac{(n-1)s_i}{2}\right] x_i + \sum_{j\in N\setminus\{i\}} \theta_j S_i(\sigma^*(x_i,\theta_{-i})).^{13}$$

4.3. **Expected cost bounds.** The expected cost bounds (ECB) requires that the utility of each agent is no less than the expected cost of the agent associated with random arrival where each arrival order is equally likely. Formally, ECB requires the following property: For any agent $i \in N$ and any profile $\theta \in \Theta^n$, $u_i(\sigma(\theta), \tau_i(\theta); \theta_i) \geq -\theta_i \left(\sum_{\sigma\in\Sigma} \frac{S_i(\sigma)}{n!}\right)$. Define $\bar{S}_i := s_i + \sum_{j\in N\setminus\{i\}}(s_j/2)$ for each $i \in N$. It is quite easy to verify that for each agent $i \in N$, $\sum_{\sigma\in\Sigma} \frac{S_i(\sigma)}{n!} = \bar{S}_i$.[14] Therefore, an equivalent representation of the ECB requirement is that for any agent $i \in N$ and any profile $\theta \in \Theta^n$, $u_i(\sigma(\theta), \tau_i(\theta); \theta_i) \geq -\theta_i\bar{S}_i$.

For a sequencing problem $\Omega \in \mathcal{S}(N)$ with acceptable utility bounds given the ECB conditions, the acceptable utility bounds vector is $O^{\bar{S}}(N,s) = (O_1^{\bar{S}_1}(s), \ldots, O_n^{\bar{S}_n}(s)) \in \mathbb{R}^n_{++}$ where for each $i \in N$, $O_i^{\bar{S}_i}(s) = \bar{S}_i$. Let $\mathcal{E}(N) = \{(\Omega, O^{\bar{S}}(N,s)) \mid \Omega \in \mathcal{S}(N)\}$ denote the set of all sequencing problems with ECB and let $\Gamma^{\bar{S}}$ represent a typical sequencing problem with ECB in $\mathcal{E}(N)$. All sequencing problem with ECB as its acceptable utility bounds satisfy the constrained acceptability property. In particular, observe that for any $\Gamma^{\bar{S}} \in \mathcal{E}(N)$ and any $i \in N$, $O_i^{\bar{S}_i}(s) = \bar{S}_i = s_i + \sum_{j\in N\setminus\{i\}}(s_j/2) > s_i$ implying that the constrained acceptability property given by condition (4) holds. Further, $\mathcal{D}(s) = \sum_{j\in N} s_j\{(s_j + A(s))/2 - (s_j + A(s))/2\} = 0$ and hence condition (8) also holds.

[13]Observe that for any $i \in N$, any $\theta_{-i} \in \Theta^{n-1}$ and any $x_i \in \mathbb{R}_+$,

$$\{S_i(\sigma^*(x_i,\theta_{-i})) - O_i(s)\}x_i = \left[\sum_{j\in P_i(\sigma^*(x_i,\theta_{-i}))} s_j + s_i - \frac{(n+1)s_i}{2}\right] x_i = \left[\sum_{j\in P_i(\sigma^*(x_i,\theta_{-i}))} s_j - \frac{(n-1)s_i}{2}\right] x_i.$$

[14]The equality $\sum_{\sigma\in\Sigma} \frac{S_i(\sigma)}{n!} = \bar{S}_i$ states that the average completion time of each agent i equals $\bar{S}_i$. The sum in $\bar{S}_i$ has two components-own processing time s_i and half of the total processing time of all other agents $j \neq i$. In any possible ordering $\sigma \in \Sigma$, an agent will always incur his own processing time and hence s_i enters $\bar{S}_i$ with probability one. Moreover, observe that any other agent $j \neq i$ precedes agent i in any ordering σ if and only if he does not precede agent i in the complement ordering σ^c. Therefore, when we consider all possible orderings to calculate agent i's average completion time, s_j for $j \neq i$ will occur in exactly half of the cases as a part of the completion time of agent i.

Therefore, $\mathcal{E}(N) \subset \overline{\mathcal{G}}(N)$. One can verify that the special feature of the relative pivotal mechanisms in this context is that the function $T_i(x_i;\theta_{-i})$ (in condition (5)) has the following form:

(14)

$$T_i^E(x_i;\theta_{-i}) = \left[\sum_{k\in P_i(\sigma^*(x_i,\theta_{-i}))} \frac{s_k}{2} - \sum_{k\in F_i(\sigma^*(x_i,\theta_{-i}))} \frac{s_k}{2}\right] x_i + \sum_{j\in N\setminus\{i\}} \theta_j S_i(\sigma^*(x_i,\theta_{-i})).$$ [15]

Remark 3. Clearly, the bounds associated with ICB and ECB are different for any sequencing problem which is not a queueing problem, that is, for any $\Omega \in \mathcal{S}(N) \setminus \mathcal{Q}(N)$. However, for any queueing problem $\Omega \in \mathcal{Q}(N)$ with $s_1 = \ldots = s_n = a > 0$, $\bar{S}_i = (n+1)a/2$ for all $i \in N$ implying that the notions of ICB and ECB are equivalent.

4.4. **Feasibility and budget balance.**

4.4.1. *Sequencing with given initial order.* Using Proposition 1 it follows that if we consider any two agent sequencing problem with initial order $(\Omega, O^{\sigma^0}(N,s)) \in \mathcal{I}(N)$, then we cannot find a mechanism that satisfies outcome efficiency, strategyproofness, acceptable utility bounds and feasibility since for any agent (i say) having first position in the initial order σ^0, $O_i(s) = s_i < A(s)$. The discussion to follow shows that this impossibility result holds in general for any sequencing problems with given initial order.

Remark 4. Consider any $\Gamma^0 = (\Omega, O^{\sigma^0}(N,s)) \in \mathcal{I}(N)$ such that $|N| \geq 3$. We provide certain observations about the minimal relative mechanism $\hat{\mu} = (\sigma^*, \hat{\tau})$ with the $T_i(x_i;\theta_{-i})$ function given by condition (12).

(IO1) Let $i \in N$ be that agent having first queueing position under that initial order σ^0, that is, $S_i(\sigma^0) = s_i$. Then, for any profile $\theta \in \Theta^n$, $\theta_i^* = s_i.\{\max\{\theta_j/s_j\}_{j\in N\setminus\{i\}}\}$ is a solution to the maximization of the function $T_i^I(x_i : \theta_{-i})$ and we select $\sigma^*(\theta_i^*,\theta_{-i})$ such that $P_i(\sigma^*(\theta_i^*,\theta_{-i})) = P_i(\sigma^0) = \emptyset$. Therefore, we have $\theta_i^*[S_i(\sigma^*(\theta_i^*,\theta_{-i})) - O_i(s)] = \theta_i^*[s_i - s_i] = 0$ and

[15]Observe that for any $i \in N$, any $\theta_{-i} \in \Theta^{n-1}$ and any $x_i \in \mathbb{R}_+$, $\{S_i(\sigma^*(x_i,\theta_{-i})) - O_i(s)\}x_i = \left[\sum_{j\in P_i(\sigma^*(x_i,\theta_{-i}))} s_j + s_i - \sum_{j\in N\setminus\{i\}} \frac{s_j}{2} - s_i\right] x_i = \left[\sum_{k\in P_i(\sigma^*(x_i,\theta_{-i}))} \frac{s_k}{2} - \sum_{k\in F_i(\sigma^*(x_i,\theta_{-i}))} \frac{s_k}{2}\right] x_i.$

hence using (12) it follows that the transfer associated with the minimal relative pivotal mechanism $\hat{\mu} = (\sigma^*, \hat{\tau})$ for agent $i \in N$ is

$$\hat{\tau}_i(\theta) = s_i \sum_{j \in P_i(\sigma^*(\theta))} \theta_j.$$

(IO2) Let $k \in N$ be that agent having last queueing position under that initial order σ^0, that is, $S_i(\sigma^0) = A(s) = \sum_{j \in N} s_j$. Then, using argument similar to the one used in (R1), it follows that for any $\theta \in \Theta^n$, $\theta^*_k = 0$ and $P_k(\sigma^*(0, \theta_{-k})) = P_i(\sigma^0) = N \setminus \{k\}$. Therefore, we have $\theta^*_i[S_i(\sigma^*(0_i, \theta_{-i})) - O_i(s)] = \theta^*_i[A(s) - A(s)] = 0$ and hence using (12) it follows that the transfer associated with the minimal relative pivotal mechanism $\hat{\mu} = (\sigma^*, \hat{\tau})$ for agent $k \in N$ is

$$\hat{\tau}_k(\theta) = -s_k \sum_{j \in F_k(\sigma^*(\theta))} \theta_j.$$

Points (IO1) and (IO2) of Remark 4 show that given a sequencing problem with initial order σ^0, the explicit form of the minimal relative pivotal transfers of the agents having the first and last positions under the initial order σ^0 are easy to derive. However, it is difficult to get an explicit form of the minimal relative pivotal transfers for agents having other positions under the initial order σ^0. Despite this difficulty, using points (IO1) and (IO2) of Remark 4 and by appropriate construction of a profile we can prove the following impossibility result.

Proposition 3. For any $\Gamma^0 = (\Omega, O^{\sigma^0}(N, s)) \in \mathcal{I}(N)$ with $|N| \geq 3$, there is no mechanism that satisfies outcome efficiency, strategyproofness, acceptable utility bounds and feasibility.

4.4.2. *ICB and ECB.* Using Proposition 1 one can show that if we consider $(\Omega, O^s(N, s)) \in \mathcal{C}(N)$ with two agents $N = \{1, 2\}$, then we cannot find a mechanism that satisfies outcome efficiency, strategyproofness, acceptable utility bounds and feasibility since we require $3s_1/2 \geq A(s)$ and $3s_2/2 \geq A(s)$ to hold simultaneously which is impossible. Similarly, using Proposition 1 one can also show that if we consider $(\Omega, O^{\bar{S}}(N, s)) \in \mathcal{E}(N)$ with two agents $N = \{1, 2\}$, then we cannot find a mechanism that satisfies outcome efficiency, strategyproofness, acceptable utility bounds and feasibility since, for each $i, j \in \{1, 2\}$ with $i \neq j$, we have $s_i + s_j/2 < A(s) = s_1 + s_2$. What happens when we have more that two agents?

Proposition 4. For any $(\Omega, O(N, s)) \in \mathcal{C}(N) \cup \mathcal{E}(N)$ such that $|N| = 3$, if we can find a feasible relative pivotal mechanism, then $\Omega \in \mathcal{Q}(N)$.

Proposition 4 states that when there are three agents, if we can find a mechanism satisfying outcome efficiency, strategyproofness, feasibility and, either ICB or ECB, then we must have a queueing problem. It is well-known from the existing literature on queueing problems that, when there are three or more agents we can find mechanisms that satisfy budget balance along with outcome efficiency, strategyproofness and ICB (or ECB).[16] Therefore, before concluding, we analyze queueing problems with acceptable utility bounds in greater details.

5. QUEUEING PROBLEMS

Throughout this section we assume without loss of generality that $s_1 = \ldots = s_n = 1$, and, given any queueing problem $\Omega \in \mathcal{Q}(N)$, we define the acceptable utility bounds vector as $O(N) = (O_1, \ldots, O_n) \in \mathbb{R}^n$. Therefore, we represent any queueing problem with acceptable utility bounds as $\Gamma^Q = (\Omega, O(N))$. Any $\Gamma^Q = (\Omega, O(N))$ satisfies the constrained acceptability property if $O(N) = (O_1, \ldots, O_n)$ is such that $O_i \geq 1$ for all $i \in N$. One can easily verify that the special feature of the relative pivotal mechanisms in this context is that the function $T_i(x_i; \theta_{-i})$ (given by (5)) has the following form:

$$T_i^Q(x_i; \theta_{-i}) = [\sigma_i^*(x_i, \theta_{-i}) - O_i]\, x_i + \sum_{j \in N \setminus \{i\}} \sigma_j^*(x_i, \theta_{-i})\theta_j. \tag{15}$$

For any queueing problem $\Omega \in \mathcal{Q}(N)$ with the acceptability parameter vector associated with either ICB or ECB is $O^B(N) = (O_1^B, \ldots, O_n^B)$ where $O_i^B = \frac{n+1}{2}$ for all $i \in N$ (see Remark 3). Given (15) we get that the function $T_i^Q(x_i; \theta_{-i})$ has the following form:

$$T_i^{QB}(x_i; \theta_{-i}) = \left[\sigma_i^*(x_i, \theta_{-i}) - \frac{(n+1)}{2}\right] x_i + \sum_{j \in N \setminus \{i\}} \sigma_j^*(x_i, \theta_{-i})\theta_j. \tag{16}$$

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[16]See Chun and Mitra [4], Chun, Mitra and Mutuswami [6] and Kayi and Ramaekers [25] for a detailed discussions on symmetrically balanced VCG mechanisms.

The discussion to follow identifies the explicit forms of the relative pivotal mechanisms.

Definition 10. For σ^* and for any positive integer $K \leq |N|$, a mechanism $\mu^k = (\sigma^*, \tau^{(K)})$ is a *K-pivotal mechanism* if for any $\theta \in \Theta^n$ and any $i \in N$,

$$\tau_i^{(K)}(\theta) = \begin{cases} -\sum\limits_{j:\sigma_i^*(\theta)<\sigma_j^*(\theta)\leq K} \theta_j & \text{if } \sigma_i^*(\theta) < K, \\ 0 & \text{if } \sigma_i^*(\theta) = K, \\ \sum\limits_{j:K\leq\sigma_j^*(\theta)<\sigma_i^*(\theta)} \theta_j & \text{if } \sigma_i^*(\theta) > K. \end{cases} \tag{17}$$

See Mitra and Mutuswami [30] who introduce and characterize the K-pivotal mechanisms for the queueing problems. Chun and Yengin [11] also provide another characterization of these mechanism. We define a new set of mechanisms which are obtained by appropriately mixing different K-pivotal mechanisms.

Definition 11. For any queueing problem, a mechanism $\bar{\mu}^a = (\sigma^*, \bar{\tau}^a)$ is a *centered K-pivotal mechanism with non-negative intercepts* if for all $\theta \in \Theta^n$ and all $i \in N$,

$$\bar{\tau}_i^a(\theta) = H_i(\theta_{-i}) + \begin{cases} \tau_i^{\left(\frac{n+1}{2}\right)}(\theta) & \text{if } n \text{ is odd}, \\ \frac{1}{2}\tau_i^{\left(\frac{n}{2}\right)}(\theta) + \frac{1}{2}\tau_i^{\left(\frac{n}{2}+1\right)}(\theta) & \text{if } n \text{ is even}, \end{cases} \tag{18}$$

where for each $i \in N$, the function $H_i : \Theta^{|N\setminus\{i\}|} \to \mathbb{R}_+$.

Corollary 1. For any queueing problem $\Omega \in \mathcal{Q}(N)$, a mechanisms satisfies outcome efficiency, strategyproofness and ICB (ECB) if and only if it is a centered K-pivotal mechanism with non-negative intercepts.

Corollary 1 generalizes a result by Chun and Yengin [11] on outcome efficienct, strategyproofness and ICB (ECB) by eliminating the gap between their necessary and sufficient conditions.

5.0.3. *Symmetrically balanced VCG mechanism.* The symmetrically balanced VCG mechanism is defined for any queueing problem with three or more agents as follows.

Definition 12. Assume $|N| \geq 3$. The mechanism $\mu^S = (\sigma^*, \tau^S)$ is the *symmetrically balanced VCG mechanism* if for all profiles $\theta \in \Theta^n$ and all $i \in N$,

$$\tau_i^S(\theta) = \sum_{j \in P_i(\sigma^*(\theta))} \left(\frac{\sigma_j^*(\theta) - 1}{n-2} \right) \theta_j - \sum_{j \in F_i(\sigma^*(\theta))} \left(\frac{n - \sigma_j^*(\theta)}{n-2} \right) \theta_j. \tag{19}$$

From the existing literature on queueing problems it is well known that the symmetrically balanced VCG mechanism satisfies outcome efficiency, strategyproofness and ICB (ECB) when there are three or more agents (see Chun and Mitra [4], Chun, Mitra and Mutuswami [6] and Kayi and Ramaekers [25]). Given Corollary 1 it means that the symmetrically balanced VCG mechanism is a centered K-pivotal mechanism with non-negative intercept when there are three or more agents. Given more than two agents, consider that centered K-pivotal mechanism with non-negative intercept for which the $H_i : \Theta^{|N \setminus \{i\}|} \rightarrow \mathbb{R}_+$ function for any $i \in N$ and any $\theta_{-i} \in \Theta^{|N| \setminus \{i\}}$ has the following form:

$$H_i(\theta_{-i}) = \begin{cases} \sum_{k=1}^{\frac{n}{2}-1} \left(\frac{k-1}{n-2}\right) \left\{\theta_{(k)}(\theta_{-i}) - \theta_{(n-k)}(\theta_{-i})\right\} & \text{if } n \text{ is even and } n \geq 4, \\ \sum_{k=1}^{\frac{n-1}{2}} \left(\frac{k-1}{n-2}\right) \left\{\theta_{(k)}(\theta_{-i}) - \theta_{(n-k)}(\theta_{-i})\right\} & \text{if } n \text{ is odd and } n \geq 3 \end{cases} \tag{20}$$

where for any $k \in \{1, \ldots, n-1\}$, $\theta_{(k)}(\theta_{-i})$ is the k-th ranked waiting cost from the profile $\theta_{-i} \in \Theta^{|N \setminus \{i\}|}$ so that $\theta_{(1)}(\theta_{-i}) \geq \ldots \geq \theta_{(n-1)}(\theta_{-i})$. One can verify that with the $H_i : \Theta^{|N \setminus \{i\}|} \rightarrow \mathbb{R}_+$ function given by (20), the resulting centered K-pivotal mechanism with non-negative intercept is the symmetrically balanced VCG mechanism.

5.1. **Feasibility and budget balance.** From Proposition 1 it follows if there are two agents, then for a queueing problem $\Omega \in \mathcal{Q}(\{1,2\})$ with acceptable utility bounds $O(\{1,2\}) = (O_1, O_2)$ we can find a mechanism satisfying outcome efficiency, strategyproofness, acceptable utility bounds and feasibility if and only if $O_1 \geq 2$ and $O_2 \geq 2$.

From Lemma 1 it follows that for any queueing problem we can find mechanisms satisfying outcome efficiency, acceptable utility bounds and feasiblity only if condition (8) holds. Condition (8) for any queueing problem reduces to the following inequality: $\sum_{j \in N} O_i/n \geq (n+1)/2$ (see Remark 1(i)). This inequality requires that the average of the acceptable utility bounds of the agents should be no less than $(n+1)/2$. The next result shows that if the acceptable utility bound of every agent is no less than $(n+1)/2$, then we can find mechanisms that satisfy outcome efficiency, strategyproofness, acceptable utility bounds and budget balance.

Proposition 5. For any $\Gamma^Q = (\Omega, O(N))$ with $|N| \geq 3$ and $O_i \geq \frac{n+1}{2}$ for all $i \in N$, we can find mechanisms that satisfy outcome efficiency, strategyproofness, acceptable utility bounds and budget balance.

To prove Proposition 5, we make use of the fact that for any queueing problem with three or more agents, the symmetrically balanced VCG mechanism satisfies outcome efficiency, strategyproofness, ICB (ECB) and, more importantly, budget balance (see Chun and Mitra [4], Chun, Mitra and Mutuswami [6] and Kayi and Ramaekers [25]). Given Remarks 1 (i), it also follows that if all agents have identical O_i's, that is, $O_i = B^*$ for all $i \in N$, then condition $O_i = B^* \geq \frac{n+1}{2}$ for all $i \in N$ is both necessary and sufficient for getting mechanisms that satisfy outcome efficiency, strategyproofness, acceptable utility bounds and budget balance.

6. CONCLUSIONS AND FURTHER WORK

Our main contribution to the literature is the introduction of a general lower bound which restricts the maximum disutility of any agent. The name "acceptable utility bounds" is self-explanatory and can be justified because it guarantees an acceptable or an appropriate level of minimum satisfaction to an agent waiting in the queue to avail a service. A customer oriented facility would always aim at enhancing satisfactory customer experience. Our bound acts as an assurance to agents and has a broad conceptual appeal. It is also shown to be compatible with the standard desirable properties in the literature. Applying this lower bound to a more generic setup, with multiple service facilities or allowing agents to arrive randomly, would serve as an interesting extension to our problem.

7. APPENDIX

Proof of Theorem 1: (SPC1) $\Rightarrow$ (SPC2). It is well-known that for an outcome efficient sequencing rule a mechanisms is strategyproof if and only if the associated transfer is a VCG transfer (see Holmström [24]). The standard way of specifying the VCG transfers for any sequencing problem Ω is that for all $\theta \in \Theta^n$ and for all $i \in N$, $\tau_i(\theta) = -C(\sigma^*(\theta)), \theta) + \theta_i S_i(\sigma^*(\theta)) + g_i(\theta_{-i})$, where for each $i \in N$ the function $g_i : \Theta^{|N\setminus\{i\}|} \to \mathbb{R}$ is arbitrary.[17] If in addition we require acceptable utility bounds to be met, then it is necessary that for any profile $\theta \in \Theta^N$ and any agent $i \in N$, $U_i(\sigma^*(\theta), \tau_i(\theta); \theta_i) = -C(\sigma^*(\theta); \theta) + g_i(\theta_{-i}) \geq -\theta_i O_i(s)$ implying that $g_i(\theta_{-i}) \geq C(\sigma^*(\theta); \theta) - \theta_i O_i(s)$. Since the function $g_i(\theta_{-i})$ is independent of agent i's waiting cost θ_i, we have the following:

$$g_i(\theta_{-i}) \geq \bar{g}_i(\theta_{-i}) := \sup_{x_i \in \Theta} [T_i(x_i; \theta_{-i})], \;\; T_i(x_i; \theta_{-i}) := [C(\sigma^*(x_i, \theta_{-i}); x_i, \theta_{-i}) - x_i O_i(s)]. \tag{21}$$

Observe that $T_i(x_i; \theta_{-i}) = [S_i(\sigma^*(x_i, \theta_{-i})) - O_i(s)]x_i + \sum_{j \in N\setminus\{i\}} \theta_j S_j(\sigma^*(x_i, \theta_{-i}))$.

Consider any profile $\tilde{\theta} \in \Theta^n$ and any $i \in N$ such that $\tilde{\theta}_j / s_j = a > 0$ for all $j \in N \setminus \{i\}$. Consider any $x'_i, x''_i \in \Theta$ such that $x'_i/s_i \geq a \geq x''_i/s_i$ and $x'_i > x''_i$. If $O_i(s) < s_i$, then we have

$$T_i(x'_i; \tilde{\theta}_{-i}) - T_i(x''_i; \tilde{\theta}_{-i}) = (x'_i - x''_i)[s_i - O_i(s)] + \sum_{j \neq i} s_i s_j \left[\frac{\tilde{\theta}_j}{s_j} - \frac{x''_i}{s_i}\right] > 0. \tag{22}$$

Moreover, for any $x_i > s_i a$, $T_i(x_i; \tilde{\theta}_{-i}) = x_i[s_i - O_i(s)] + \sum_{j \in N\setminus\{i\}} \tilde{\theta}_j S_j(\sigma^*(x_i, \tilde{\theta}_{-i}))$ is increasing in x_i. Therefore, the x^*_i that maximizes $T_i(x_i; \tilde{\theta}_{-i})$ is then $x^*_i = \infty$ implying that we do not have a supremum. Therefore, for a supremum to exist it is necessary that $O_i(s) \geq s_i$.

(SPC2) $\Rightarrow$ (SPC1). Consider any Γ that satisfies the constrained acceptability property, that is, consider $\Gamma \in \mathcal{G}(N)$. For any profile $\theta \in \Theta^n$ and any $i \in N$, consider the type $x^*_i \in \Theta$ such that it is a supremum for the function $T_i(x_i, \theta_{-i})$.

Step 1: For any $i \in N$ and any $\theta_{-i} \in \Theta^{|N\setminus\{i\}|}$, there exists $x^*_i \in \{\{s_i(\theta_k/s_k)\}_{k \in N\setminus\{i\}} \cup \{0\}\}$ such that $T_i(x^*_i; \theta_{-i}) \geq T_i(x_i; \theta_{-i})$ for all $x_i \in \Theta$.

Proof of Step 1: Consider any agent $i \in N$ and any $\theta_{-i} \in \Theta^{|N\setminus\{i\}|}$ and we define the vector $\tilde{R}(\theta_{-i}) = ((\tilde{R}_j(\theta_{-i}) = \theta_j/s_j))_{j \neq i})$ of agent specific waiting cost to processing time ratio of agents in $N \setminus \{i\}$ and $R(\theta_{-i}) = (R_1(\theta_{-i}) =$

[17] See Mitra [28] and Suijs [37].

$\theta_{(1)}/s_{(1)}, \ldots, R_{n-1}(\theta_{-i}) = \theta_{(n-1)}/s_{(n-1)})$ be the permutation of $\tilde{R}(\theta_{-i})$ such that $R_1(\theta_{-i}) \geq \ldots \geq R_{n-1}(\theta_{-i})$. We divide the proof into two possibilities (a) $O_i(s) \in [s_i, A(s)]$ and (b) $O_i(s) > A(s)$.

Proof of Possibility (a): We first show that there exists $x_i^* \in [s_i R_{n-1}(\theta_{-i}), s_i R_1(\theta_{-i})]$ that maximizes $T_i(x_i, \theta_{-i})$. Observe that for any $x_i \in \Theta$, the function $T_i(x_i;\theta_{-i}) = [S_i(\sigma^*(x_i, \theta_{-i})) - O_i(s)]x_i + \sum_{j \in N\setminus\{i\}} \theta_j S_j(\sigma^*(x_i, \theta_{-i}))$. If $x_i > s_i R_1(\theta_{-i})$, then $S_i(\sigma^*(x_i, \theta_{-i})) = s_i$ and hence $T_i(x_i;\theta_{-i}) = [s_i - O_i(s)]x_i + \sum_{j \in N\setminus\{i\}} \theta_j S_j(\sigma^*(x_i, \theta_{-i}))$ which is non-increasing in x_i since by interval property $s_i \leq O_i(s)$ implying that the coefficient of x_i in $T_i(x_i;\theta_{-i})$ is non-positive. Hence, (i) if a maxima exists then we can always find a waiting cost $x_i^* \leq s_i R_1(\theta_{-i})$ that achieves it. Similarly, if $y_i < s_i R_{n-1}(\theta_{-i})$, then $S_i(\sigma^*(y_i, \theta_{-i})) = A(s)$ and hence it follows that $T_i(y_i;\theta_{-i}) = [A(s) - O_i(s)]y_i + \sum_{j \in N\setminus\{i\}} \theta_j S_i(\sigma^*(y_i, \theta_{-i}))$ which is non-decreasing in y_i since by interval property $A(s) \geq O_i(s)$ implying that the coefficient of x_i in $T_i(x_i;\theta_{-i})$ is non-negative. Hence, (ii) if a maxima exists, then we can always find a waiting cost $x_i^* \geq s_i R_{n-1}(\theta_{-i})$ that achieves it.

The function $T_i(x_i;\theta_{-i})$ is continuous and concave in x_i on the interval $[s_i R_{n-1}(\theta_{-i}), s_i R_1(\theta_{-i})]$ and the interval $[s_i R_{n-1}(\theta_{-i}), s_i R_1(\theta_{-i})]$ is compact.[18] Hence, the function $T_i(x_i;\theta_{-i})$ has a maxima in the interval $[s_i R_{n-1}(\theta_{-i}), s_i R_1(\theta_{-i})]$. Given $x_i^* \in [s_i R_{n-1}(\theta_{-i}), s_i R_1(\theta_{-i})]$ and given continuity of $T_i(x_i;\theta_{-i})$, for two agents the proof is complete since $x_i^* = s_i R_1(\theta_j) = s_i(\theta_j/s_j)$ and it follows that $T_i(\theta_i(\theta_j), \theta_j) = [s_i - O_i(s)]s_i(\theta_j/s_j) + \theta_j(s_i + s_j)$. Therefore, consider the more than two agents case. If there exists $k \in N\setminus\{i\}$ such that $x_i^* = s_i(\theta_{(k)}/s_{(k)})$ (so that $T_i(x_i^*;\theta_{-i}) = T_i(s_i(\theta_k/s_k);\theta_{-i}) \geq T_i(x_i;\theta_{-i})$ holds for all $x_i \in \Theta$), then the proof is complete. If not then suppose there exists

[18]From the functional form of $T_i(x_i;\theta_{-i})$ and given outcome efficiency it is obvious that given any θ_{-i}, the function $T_i(x_i;\theta_{-i})$ is continuous in x_i on any open interval $(s_i R_{k+1}(\theta_{-i}), s_i R_k(\theta_{-i}))$ for all $k \in \{1, \ldots, n-2\}$ and by using appropriate limit argument one can also show continuity at any point $s_i R_k(\theta_{-i})$ for $k \in \{1, \ldots, n-1\}$. For concavity note that for any $\theta_{-i} \in \Theta_{-i}$, for every $x_i \in (s_i R_{k+1}(\theta_i), s_i R_k(\theta_i))$ for all $k \in \{0, \ldots, n\}$, where $R_{n+1} = 0$ and $R_0 = \infty$, $T_i(x_i;\theta_{-i}) = [S_i(\sigma^*(x_i, \theta_i)) - O_i(s)]\, x_i + \sum_{j \in N\setminus\{i\}} \theta_j s_j(\sigma^*(x_i, \theta_i))$ is a straight line. Moreover, $S_i(\sigma^*(x_i, \theta_i))$ is non-increasing in $x_i \in \mathbb{R}_{++}$. Hence, the slope $S_i(\sigma^*(x_i, \theta_i)) - O_i(s)$ is also non-increasing for $x_i \in \mathbb{R}_{++}$. As a result the piece-wise linear continuous function $T_i(x_i;\theta_{-i})$ is concave for $x_i \in \mathbb{R}_{++}$.

$k \in \{1, \ldots, n-2\}$ such that $x_i^* \in (s_i R_{k+1}(\theta_{-i}), s_i R_k(\theta_{-i}))$, that is,

$$T_i(x_i^*; \theta_{-i}) = \left[\sum_{r=1}^{k} s_{(r)} + s_i - O_i(s) \right] x_i^* + \sum_{j \in N \setminus \{i\}} \theta_j S_j(\sigma^*(x_i^*, \theta_{-i})).$$

If $\sum_{r=1}^{k} s_{(r)} + s_i - O_i(s) > 0$, then for any $x_i \in (x_i^*, s_i R_k(\theta_{-i})]$, $\sigma^*(x_i, \theta_{-i}) = \sigma^*(x_i^*, \theta_{-i})$ and $T_i(x_i; \theta_{-i}) > T_i(x_i^*; \theta_{-i})$ since $T_i(x_i; \theta_{-i}) - T_i(x_i^*; \theta_{-i}) = \left[\sum_{r=1}^{k} s_{(r)} + s_i - O_i(s)\right](x_i - x_i^*) > 0$. Therefore we have a contradiction to our assumption that at x_i^* the function $T_i(x_i; \theta_{-i})$ is maximized. If $\sum_{r=1}^{k} s_{(r)} + s_i - O_i(s) < 0$, then for any $x_i' \in [s_i R_k(\theta_{-i}), x_i^*)$, $\sigma^*(x_i', \theta_{-i}) = \sigma^*(x_i^*, \theta_{-i})$ and $T_i(x_i'; \theta_{-i}) > T_i(x_i^*; \theta_{-i})$ since $T_i(x_i'; \theta_{-i}) - T_i(x_i^*; \theta_{-i}) = \left[\sum_{r=1}^{k} s_{(r)} + s_i - O_i(s)\right](x_i' - x_i^*) > 0$. Again we have a contradiction to our assumption that at x_i^* the function $T_i(x_i; \theta_{-i})$ is maximized. Therefore, the only possibility left is $\sum_{r=1}^{k} s_{(r)} + s_i - O_i(s) = 0$. However, in that case $T_i(x_i^*; \theta_{-i}) = \sum_{j \in N \setminus \{i\}} \theta_j S_i(\sigma^*(x_i^*, \theta_{-i}))$ and for every $x_i \in [s_i R_{k+1}(\theta_{-i}), s_i R_k(\theta_{-i})]$ the function $T_i(x_i, \theta_{-i})$ attains its maximum value implying that $T_i(x_i^*; \theta_{-i}) = T_i(s_i R_{k+1}(\theta_{-i}); \theta_{-i}) = T_i(s_i R_k(\theta_{-i}); \theta_{-i})$ and Step 1 continues to be valid.

Proof of Possibility (b): If $O_i(s) > A(s)$, then for any $i \in N$ and any given $\theta_{-i} \in \Theta^{|N \setminus \{i\}|}$, the function $T_i(x_i; \theta_{-i})$ on $\mathbb{R}_+$ is maximized if we set $x_i^* = 0$. Since the function $T_i(x_i; \theta_{-i})$ is only defined on the domain $\Theta^n = \mathbb{R}_+ \setminus \{0\}$, $x_i^* = 0$ acts as a supremum of the function $T_i(x_i; \theta_{-i})$ and that $T_i(0; \theta_{-i}) = \sum_{j \in N \setminus \{i\}} \theta_j S_j(\sigma^*(0, \theta_{-i})) > T_i(x_i; \theta_{-i})$ for all $x_i \in \Theta$.

Fix any $i \in N$. First, suppose that $O_i(s) \in [s_i, A(s)]$. Given the proof of Possibility (a) of Step 1 and given any $\theta_{-i} \in \Theta^{n-1}$, let us define $x_i^* := \theta_i^*$ so that $T_i(x_i^*; \theta_{-i}) = T_i(\theta_i^*; \theta_{-i})$ and there exists $k \in N \setminus \{i\}$ such that $\theta_i^* = s_i(\theta_k / s_k)$. Consider the VCG transfer having the following property: For all $\theta \in \Theta^n$ and for all $i \in N$, $\tau_i^*(\theta) = -C(\sigma^*(\theta); \theta) + \theta_i S_i(\sigma^*(\theta)) + \bar{g}_i(\theta_{-i})$ with $\bar{g}_i(\theta_{-i}) := T_i(\theta_i^*; \theta_{-i})$. Then for any given $\theta \in \Theta^n$ and any agent $i \in N$, we have $u_i(\mu_i^*(\theta), \theta_i) + \theta_i O_i(s) = -[S_i(\sigma^*(\theta) - O_i(s)]\theta_i + \bar{g}_i(\theta_{-i}) = T_i(\theta_i^*, \theta_{-i}) - T_i(\theta_i, \theta_{-i}) \geq 0$. The last inequality follows from the fact that $T_i(\theta_i, \theta_{-i}) \leq T_i(\theta_i^*, \theta_{-i})$ for all $\theta_i \in \Theta$. Hence, $u_i(\mu_i^*(\theta), \theta_i) \geq -\theta_i O_i(s)$ implying that this VCG transfer satisfies the acceptable utility bounds for agent i. Next, suppose that $O_i(s) > A(s)$. Given the proof of Possibility (b) of Step 1 and given any $\theta_{-i} \in \Theta^{n-1}$, let us define

$x_i^* := 0$ so that $T_i(x_i;\theta_{-i}) \leq T_i(0;\theta_{-i})$ for all $x_i \in \Theta$. Consider the VCG transfer having the following property: For all $\theta \in \Theta^n$ and for all $i \in N$, $\tau_i^*(\theta) = -C(\sigma^*(\theta);\theta) + \theta_i S_i(\sigma^*(\theta)) + \bar{g}_i(\theta_{-i})$ with $\bar{g}_i(\theta_{-i}) := T_i(0;\theta_{-i})$. Then for any given $\theta \in \Theta^n$ and any agent $i \in N$, we have $u_i(\mu_i^*(\theta), \theta_i) + \theta_i O_i(s) = -[S_i(\sigma^*(\theta) - O_i(s)]\theta_i + \bar{g}_i(\theta_{-i}) = T_i(0;\theta_{-i}) - T_i(\theta_i;\theta_{-i}) \geq 0$. Thus, using the constrained acceptability property we have identified VCG transfers that satisfies the acceptable utility bounds. □

Proof of Theorem 2: For outcome efficiency and strategyproof it is necessary that the mechanism $\mu = (\sigma^*, \tau)$ must be VCG with transfers satisfying the following property: For any profile $\theta \in \Theta^n$ and any agent $i \in N$, $\tau_i(\theta) = -C(\sigma^*(\theta);\theta) + \theta_i S_i(\sigma^*(\theta)) + g_i(\theta_{-i})$ where $g_i : \Theta^{|N\setminus\{i\}|} \to \mathbb{R}$ is arbitrary. For the acceptable utility bounds to hold, in addition, it is necessary that

(I) $g_i(\theta_{-i}) \geq \bar{g}_i(\theta_{-i}) = T_i(\theta_i^*;\theta_{-i}) \in \max_{x_i \in \Theta} T_i(x_i;\theta_{-i})$ and $T_i(x_i;\theta_{-i}) = [S_i(\sigma^*(x_i,\theta_{-i})) - O_i(s)]x_i + \sum_{j \in N\setminus\{i\}} \theta_j S_j(\sigma^*(x_i,\theta_{-i}))$ (see condition (21) in the proof of Theorem 1).

Hence, using (I) we can replace $g_i(\theta_{-i}) = h_i(\theta_{-i}) + T_i(\theta_i^*;\theta_{-i})$ where $h_i : \Theta^{|N\setminus\{i\}|} \to \mathbb{R}$ and $h_i(\theta_{-i}) \geq 0$. By substituting $g_i(\theta_{-i}) = h_i(\theta_{-i}) + T_i(\theta_i^*;\theta_{-i})$ in the transfer $\tau_i(\theta)$ and then simplifying it we get

$$\tau_i(\theta) = [S_i(\sigma^*(\theta_i^*,\theta_{-i})) - O_i(s)]\theta_i^* + \sum_{j \in N\setminus\{i\}} \theta_j \delta_{ji}(\theta) + h_i(\theta_{-i}), \tag{23}$$

where $\delta_{ji}(\theta) := \left(\sum_{k \in P_j(\sigma^*(\theta_i^*,\theta_{-i}))} s_k - \sum_{k \in P_j(\sigma^*(\theta))} s_k\right)$. Observe the following:

(a) If $P_i(\sigma^*(\theta)) = P_i(\sigma^*(\theta_i^*,\theta_{-i}))$, then for any $j \in N \setminus \{i\}$ we have $P_j(\sigma^*(\theta)) = P_j(\sigma^*(\theta_i^*,\theta_{-i}))$, then it easily follows that $\delta_{ji}(\theta) = 0 = (|P_j(\sigma^*(\theta_i^*,\theta_{-i}))| - |P_j(\sigma^*(\theta))|)s_i$.
(b) If $P_i(\sigma^*(\theta_i^*,\theta_{-i})) \subset P_i(\sigma^*(\theta))$, then for agent any $j \in P_i(\sigma^*(\theta)) \setminus P_i(\sigma^*(\theta_i^*,\theta_{-i}))$, we have $P_j(\sigma^*(\theta_i^*,\theta_{-i})) \setminus P_j(\sigma^*(\theta)) = \{i\}$. Hence, $\delta_{ji}(\theta) = s_i = (|P_j(\sigma^*(\theta_i^*,\theta_{-i}))| - |P_j(\sigma^*(\theta))|)s_i$.
(c) If $P_i(\sigma^*(\theta)) \subset P_i(\sigma^*(\theta_i^*,\theta_{-i}))$, then for any $j \in P_i(\sigma^*(\theta_i^*,\theta_{-i})) \setminus P_i(\sigma^*(\theta))$, it easily follows that $P_j(\sigma^*(\theta)) \setminus P_j(\sigma^*(\theta_i^*,\theta_{-i})) = \{i\}$. Therefore, we obtain $\delta_{ji}(\theta) = -s_i = (|P_j(\sigma^*(\theta_i^*,\theta_{-i}))| - |P_j(\sigma^*(\theta))|)s_i$.

By substituting the values of $\delta_{ji}(\theta)$ for possibilities (a), (b) and (c) in the sum $\sum_{j\in N\setminus\{i\}}\theta_j\delta_{ji}(\theta)$ of (23) we get the sum in (6).

From (I) condition (23) and the expansion of the sum $\sum_{j\in N\setminus\{i\}}\theta_j\delta_{ji}(\theta)$ summarized in (a), (b) and (c) we get $\tau=\tau^p$.

To prove the converse, observe that since any μ^p is a particular type of VCG transfers, μ^p is sufficient to ensure outcome efficency and strategyproofness. To complete the proof we need to check the sufficiency of acceptable utility bounds with μ^p. Consider any relative pivotal mechanism μ^p. For any $\theta\in\Theta^n$ and any $i\in N$, we have $u_i(\sigma^*(\theta),\tau_i^p(\theta),\theta_i)+\theta_iO_i(s)=-\theta_i[S_i(\sigma^*(\theta))-O_i(s)]+[S_i(\sigma^*(\theta_i^*,\theta_{-i}))-O_i(s)]\theta_i^*+\sum_{j\in N\setminus\{i\}}(|P_j(\sigma^*(\theta_i^*,\theta_{-i}))|-|P_j(\sigma^*(\theta))|)\theta_js_i+h_i(\theta_{-i})=T_i(\theta_i^*,\theta_{-i})-T_i(\theta)+h_i(\theta_{-i})\geq 0$. Therefore, $u_i(\mu_i^p(\theta),\theta_i)+\theta_iO_i(s)\geq 0$ implying $u_i(\mu_i^p(\theta),\theta_i)\geq-\theta_iO_i(s)$. Hence, any relative pivotal mechanism μ^p satisfies the relevant acceptable utility bounds. □

Proof of Lemma 1: Suppose $\Gamma=(\Omega,O(N,s))\in\mathcal{G}(N)$ is a problem for which we can find a mechanism that satisfies outcome efficiency, acceptable utility bounds and feasibility and let $\mu=(\sigma^*,\tau)$ be such a mechanism. Then using acceptable utility bounds it follows that for every $\theta\in\Theta^n$ and each $i\in N$, $u_i(\sigma^*(\theta),\tau(\theta);\theta_i)=-\theta_iS_i(\sigma^*(\theta))+\tau_i(\theta)\geq-\theta_iO_i(s)$ implying that for all $i\in N$, $\tau_i(\theta)\geq\theta_iS_i(\sigma^*(\theta))-\theta_iO_i(s)$. By summing the transfers over all agents and applying feasibility it follows that $C(\sigma^*(\theta);\theta)-\sum_{j\in N}\theta_jO_j(s)\leq 0$. Hence, for the mechanism $\mu=(\sigma^*,\tau)$ to satisfy outcome efficiency, acceptable utility bounds and feasibility it is necessary that

$$\sum_{j\in N}\theta_j\left\{O_j(s)-S_j(\sigma^*(\theta))\right\}\geq 0,\ \forall\ \theta\in\Theta^n. \tag{24}$$

Consider a set of profiles, $\theta^t=(\theta_1^t,\dots,\theta_n^t)\in\Theta^n$ defined for any positive integer t such that $\theta_j^t=s_j[1-\{j/(2^tn)\}]$ for all $j\in N$. Observe that for any given t and any $l,m\in N$ such that $l<m$, $\theta_l^t/s_l>\theta_m^t/s_m$ so that for every positive integer t, we have the same outcome efficient order $\sigma^*(\theta^t)=(\sigma_1^0,\dots,\sigma_n^0)$ with $\sigma_j^0=j$ for all $j\in N$. Also observe that as $t\to\infty$, $\theta_j^t\to s_j>0$. Given (24), the condition $\sum_{j\in N}\theta_j^t\left\{O_j(s)-S_j(\sigma^0)\right\}\geq 0$ must hold for every positive integer t and hence it must also hold at the limiting value of t as well, that is, it must also hold when

$\theta_j = s_j$ for all $j \in N$. Hence, it is also necessary that

$$\sum_{j \in N} s_j \left\{ O_j(s) - S_j(\sigma^0) \right\} \geq 0. \tag{25}$$

If we can show that the equality $\sum_{j \in N} s_j S_j(\sigma^0) = \sum_{j \in N} s_j \{s_j + A(s)\}/2$ holds, then one can easily verify that using this equality in (25) we get the result.[19] Hence, our final step is to show this equality. Observe that

$$\begin{aligned} \sum_{j \in N} s_j S_j(\sigma^0) &= \sum_{j \in N} s_j \left(s_j + \sum_{k > j} s_k \right) = \sum_{j \in N} s_j^2 + \sum_{j \in N} \sum_{k > j} s_j s_k \\ &= \sum_{j \in N} s_j^2 + \sum_{j \in N} \left(\sum_{k \neq j} \frac{s_j s_k}{2} \right) = \sum_{j \in N} s_j \left(s_j + \sum_{k \neq j} \frac{s_k}{2} \right) \\ &= \sum_{j \in N} s_j \left(\frac{2 s_j + \sum_{k \neq j} s_k}{2} \right) = \sum_{j \in N} s_j \left(\frac{s_j + A(s)}{2} \right). \end{aligned} \tag{26}$$

Therefore, from (26) we get the required equality and the result follows. □

Proof of Proposition 1: Consider any $\Gamma = (\Omega, O(N, s)) \in \mathcal{G}(N)$ with $N = \{1, 2\}$ and, given constrained acceptability property assume without loss of generality that $O_1(s) = s_1 + \lambda_1 s_2$ and $O_2(s) = s_2 + \lambda_2 s_1$ where $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$. If $\theta = (\theta_1, \theta_2) \in \Theta^2$ is any profile such that $\theta_1/s_1 > \theta_2/s_2$, then, given $\theta_i^* = s_i \theta_j / s_j$ if $\lambda_i \in [0, 1)$ and $\theta_i^* = 0$ if $\lambda_i \geq 1$ for any $i, j \in \{1, 2\}$ such that $i \neq j$, from the definition of minimal relative pivotal mechanism $\hat{\mu}^p = (\sigma^*, \hat{\tau}^p)$ it follows that

$$\hat{\tau}_1^p(\theta_1, \theta_2) = -\min\{\lambda_1, 1\} \theta_2 s_1 \text{ and } \hat{\tau}_2^p(\theta_1, \theta_2) = (1 - \min\{\lambda_2, 1\}) \theta_1 s_2. \tag{27}$$

Therefore, from (27) it follows that

$$\hat{\tau}_1^p(\theta_1, \theta_2) + \hat{\tau}_2^p(\theta_1, \theta_2) = \left[(1 - \min\{\lambda_2, 1\}) \theta_1 s_2 - \min\{\lambda_1, 1\} \theta_2 s_1 \right]. \tag{28}$$

Feasibility requires that $\hat{\tau}_1^p(\theta_1, \theta_2) + \hat{\tau}_2^p(\theta_1, \theta_2) \leq 0$ for all $\theta = (\theta_1, \theta_2) \in \Theta^2$ and for any θ_1 and any θ_2 such that $\theta_1/s_1 > \theta_2/s_2$, (I) $(1 - \min\{\lambda_2, 1\}) \theta_1 s_2 \leq$

[19] Specifically, if $\sum_{j \in N} s_j S_j(\sigma^0) = \sum_{j \in N} s_j \{s_j + A(s)\}/2$, then expanding the left hand side of (25) we get

$$\sum_{j \in N} s_j O_j(s) - \sum_{j \in N} s_j S_j(\sigma^0) = \sum_{j \in N} s_j O_j(s) - \sum_{j \in N} s_j \left(\frac{s_j + A(s)}{2} \right) = \sum_{j \in N} s_j \left\{ O_j(s) - \left(\frac{s_j + A(s)}{2} \right) \right\}.$$

$\min\{\lambda_1, 1\}\theta_2 s_1$. If $(1 - \min\{\lambda_2, 1\}) > 0$ (that is, if $\lambda_2 \in [0, 1)$), then given any $\theta_2 > 0$ and any $\lambda_1 \geq 0$, by taking any θ_1 sufficiently large such that $\theta_1 > \min\{\lambda_1, 1\}s_1\theta_2/(1 - \min\{\lambda_2, 1\})s_2$ and making it sufficiently large we have a violation of condition (I). Hence, $\lambda_2 \geq 1$. Similarly, if $\theta' = (\theta'_1, \theta'_2) \in \Theta^2$ is such that $\theta'_1/s_1 < \theta'_2/s_2$, then, given $\lambda_2 \geq 1$, from the definition of minimal relative pivotal mechanism $\hat{\mu}^p = (\sigma^*, \hat{\tau}^p)$ it follows that

$$\hat{\tau}^p_1(\theta'_1, \theta'_2) = (1 - \min\{\lambda_1, 1\})\theta'_2 s_1 \text{ and } \hat{\tau}^p_2(\theta'_1, \theta'_2) = -\theta'_1 s_2. \tag{29}$$

Feasibility requires that $\hat{\tau}^p_1(\theta'_1, \theta'_2) + \hat{\tau}^p_2(\theta'_1, \theta'_2) \leq 0$ for all $\theta' = (\theta'_1, \theta'_2) \in \Theta^2$ and hence given (29) for any θ'_1 and any θ'_2 such that $\theta'_1/s_1 < \theta'_2/s_2$, for feasibility it is necessary that (II) $(1 - \min\{\lambda_1, 1\})\theta'_2 s_1 \leq \theta'_1 s_2$. If $(1 - \min\{\lambda_1, 1\}) > 0$ (that is, $\lambda_1 \in [0, 1)$), then given any θ'_1, by taking $\theta'_2 > s_2\theta'_1/(1 - \min\{\lambda_1, 1\})s_1$ we have a violation of condition (II). Hence, we must also have $\lambda_1 \geq 1$. Therefore, for feasibility it is necessary that $\lambda_1 \geq 1$ and $\lambda_2 \geq 1$, that is, $O_1(s) \geq A(s)$ and $O_2(s) \geq A(s)$.

Conversely, if $\lambda_1 \geq 1$ and $\lambda_2 \geq 1$, then, from the definition of minimal relative pivotal mechanism $\hat{\mu}^p = (\sigma^*, \hat{\tau}^p)$, it follows that for any $\theta \in \Theta^2$, any $i \in \{1, 2\}$ and any $j \in \{1, 2\}$ with $j \neq i$,

$$\hat{\tau}^p_i(\theta) = \begin{cases} -\theta_j s_i & \text{if } P_i(\sigma^*(\theta)) = \emptyset, \\ 0 & \text{if } P_i(\sigma^*(\theta_i(\theta_{-i}), \theta_{-i}) = \{j\}, \end{cases} \tag{30}$$

It is immediate from (30) that for all $\theta_1, \theta_2 \in \Theta$, then we get feasibility. Hence, we have the first part of the result.

The proof of the second part, that is, any relative pivotal mechanism given by (30) is not budget balanced, is a special case of Proposition 3 in De and Mitra [16] where we need to replace linear sequencing rule by its special case of outcome efficient sequencing rule. □

Proof of Proposition 2: Consider any $\Gamma = (\Omega, O(N, s)) \in \overline{\mathcal{G}}(N)$ with the acceptable utility bounds satisfying the following properties: $O_i(s) \geq A(s) = \sum_{j \in N} s_j$ for all $i \in N$. Observe that the constrained acceptability property given by condition (4) holds for this example as well. For any $\theta \in \Theta^n$ and any $i \in N$, the function $T_i(x_i; \theta_{-i})$ (given by Definition 7)) has a supremum at $\theta^*_i = 0$ for all $i \in N$ implying that $P_i(\sigma^*(0,,\theta_{-i})) \cup \{i\} = n$ and hence $S_i(\sigma^*(0,\theta_{-i})) = A(s) \leq O_i(s)$. The reason is the following: For any $i \in N$ and any $x_i \in \Theta$ such that $P_i(\sigma^*(x_i,\theta_{-i})) \subset N \setminus \{i\}$ and $P_i(\sigma^*(x_i,\theta_{-i})) \neq N \setminus \{i\}$, the function $T_i(x_i;\theta_{-i}))$ is decreasing in x_i since $[S_i(\sigma^*(x_i,\theta_{-i})) - O_i(s)] = \sum_{j\in P_i(\sigma^*(x_i,\theta_{-i}))} s_j - \sum_{j\in N\setminus\{i\}} s_j = -\sum_{j\in F_i(\sigma^*(x_i,\theta_{-i}))} s_j$ is negative. Therefore, for any $i \in N$, $\theta_i^* = 0$ implying that agent i is always served last in the benchmark order $\sigma^*(0,\theta_{-i})$. Given $\theta_i^* = 0$, it is quite easy to verify that (I) $\theta_i^*[S_i(\sigma^*(x_i,\theta_{-i})) - O_i(s)] = 0$ and (II) $RP_i(\theta) = -\sum_{k\in F_i(\sigma^*(\theta))} \theta_k s_i$. Therefore, using (I) and (II) in Definition 7 we get that an outcome efficient mechanism $\mu^p = (\sigma^*, \tau^p)$ is a relative pivotal mechanism if τ^p satisfies the following property: For any profile $\theta \in \Theta^n$ and any agent $i \in N$,

$$\tau_i^p(\theta) = -\sum_{k\in F_i(\sigma^*(\theta))} \theta_k s_i + h_i(\theta_{-i}), \tag{31}$$

where $h_i : \Theta^{|N\setminus\{i\}|} \to \mathbb{R}_+$. Let $n \geq 3$ and for all $i \in N$ and all $\theta_{-i} \in \Theta^{|N\setminus\{i\}|}$, suppose we set $h_i(\theta_{-i}) = \sum_{j\in N\setminus\{i\}} \left\{ s_j \sum_{k\in F_j(\sigma^*(\theta_{-i}))} \theta_k \right\} / (n-2)$ in the transfer given by (31). One can then simplify the resulting transfers (31) and show that we get budget balance.[20] □

Proof of Proposition 3: Consider any $\Gamma^0 = (\Omega, O^{\sigma^0}(N,s)) \in \mathcal{I}(N)$ and, without loss of generality, assume σ^0 such that $\sigma_i^0 = i$ for all $i \in N$. Consider any $\theta \in \Theta^n$ such that $\theta_n/s_n > \theta_1/s_1 > \ldots > \theta_{n-1}/s_{n-1}$ so that $P_1(\sigma^*(\theta)) = \{n\}$, $P_j(\sigma^*(\theta)) = \{1,\ldots,j-1\} \cup \{n\}$ for all $j \in N \setminus \{1,n\}$ and $P_n(\sigma^*(\theta)) = \emptyset$. Consider the minimal relative pivotal mechanism $\hat{\mu} = (\sigma^*, \hat{\tau})$ (in Definition 9) with the $T_i(x_i;\theta_{-i})$ function given by (12). It is easy to verify the following:

(i) Given $P_1(\sigma^0) = \emptyset$, from (IO1) of Remark 4 we have $\theta_1^* = s_1\theta_n/s_n$ and $P_1(\sigma^*(\theta_1^*,\theta_{-1})) = P_1(\sigma^0) = \emptyset$. Further, $P_n(\sigma^*(\theta_1^*,\theta_{-1})) \setminus P_n(\sigma^*(\theta)) = \{1\}$ and $P_j(\sigma^*(\theta_1^*,\theta_{-1})) = P_j(\sigma^*(\theta))$ for all $j \in N \setminus \{1,n\}$. Thus, $\hat{\tau}_1(\theta) = (|P_n(\sigma^*(\theta_1^*,\theta_{-1}))| - |P_n(\sigma^*(\theta))|)\theta_n s_1 = \theta_n s_1$.

(ii) Given $P_n(\sigma^0) = N \setminus \{n\}$, from condition (IO2) of Remark 4 we get $\theta_n^* = s_n\theta_{n-1}/s_{n-1}$ and $P_n(\sigma^*(\theta_n^*,\theta_{-n})) = P_n(\sigma^0) = N \setminus \{n\}$. Moreover, $P_j(\sigma^*(\theta)) \setminus P_j(\sigma^*(\theta_n^*,\theta_{-n})) = \{n\}$ for all $j \in N \setminus \{n\}$. Hence, the

[20] We do not provide a formal proof since it is a special case of the proof of Theorem 1 in De and Mitra [16].

transfer of n is $\hat{\tau}_n(\theta) = \sum_{j\in N\setminus\{n\}}(|P_j(\sigma^*(\theta^*_n,\theta_{-n}))| - |P_j(\sigma^*(\theta))|)\theta_j s_n = -\sum_{j\in N\setminus\{n\}}\theta_j s_n$. Therefore, the transfer of agent n does not involve the waiting cost θ_n.

(iii) Finally, consider any $k \in N\setminus\{1,n\}$. Observe that if $x_k = s_k\theta_n/s_n$, then $T^I_k(x_k;\theta_{-k}))$ is decreasing in x_k since the coefficient of x_k, that is $[\sum_{j\in P_k(\sigma^*(x_k,\theta_{-k}))} s_j - \sum_{j\in P_k(\sigma^0)} s_j] = -\sum_{j=1}^{k-1} s_j < 0$. Hence, $\theta^*_k \neq s_k\theta_n/s_n$. Further, $(|P_n(\sigma^*(\theta^*_k,\theta_{-k}))| - |P_n(\sigma^*(\theta))|)\theta_n s_k = 0$ since $P_n(\sigma^*(\theta^*_k,\theta_{-k})) = P_n(\sigma^*(\theta)) = \emptyset$. Thus, the transfer of any agent $k \in N\setminus\{1,n\}$ does not involve the waiting cost θ_n of agent n and hence can be expressed in the following form: $\hat{\tau}_k(\theta) = \theta^*_k[\sum_{j\in P_k(\sigma^*(\theta^*_k,\theta_{-k}))} s_j - \sum_{j\in P_k(\sigma^0)} s_k] + \sum_{j\in N\setminus\{k,n\}}(|P_j(\sigma^*(\theta^*_k,\theta_{-k}))| - |P_j(\sigma^*(\theta))|)\theta_j s_k$.

From (i), (ii) and (iii) it follows that $\sum_{j\in N}\hat{\tau}_j(\theta) = \theta_n s_1 + \sum_{j\in N\setminus\{1\}}\hat{\tau}_j(\theta)$. From (i) and (iii) above it also follows that the sum $\sum_{j\in N\setminus\{1\}}\hat{\tau}_j(\theta)$ does not involve the waiting cost θ_n and hence by defining $\mathcal{T}(\sigma^*(\theta);\theta_{-n}) := \sum_{j\in N\setminus\{1\}}\hat{\tau}_j(\theta)$ we get

$$\sum_{j\in N}\hat{\tau}_j(\theta) = \theta_n s_1 + \mathcal{T}(\sigma^*(\theta);\theta_{-n}). \tag{32}$$

If $\sum_{j\in N}\hat{\tau}_j(\theta) > 0$, then we have a violation of feasibility and the proof is complete. Therefore, assume $\sum_{j\in N}\hat{\tau}_j(\theta) = \theta_n s_1 + \mathcal{T}(\sigma^*(\theta);\theta_{-n}) \leq 0$. Given that $\mathcal{T}(\sigma^*(\theta);\theta_{-n})$ is independent of θ_n, if we increase the waiting cost of agent n to any $y_n(> \theta_n)$ by keeping θ_{-n} fixed, then the outcome efficient order remains unchanged (that is, $\sigma^*(y_n,\theta_{-n}) = \sigma^*(\theta)$ for all $y_n > \theta_n$) and the transfers of all but agent 1 continues to remain unchanged due to above mentioned independence argument, that is, $\mathcal{T}(\sigma^*(y_n,\theta_{-n});\theta_{-n}) = \mathcal{T}(\sigma^*(\theta);\theta_{-n})$ for all $y_n > \theta_n$. Hence, we have

$$\sum_{j\in N}\hat{\tau}_j(y_n,\theta_{-n}) = y_n s_1 + \mathcal{T}(\sigma^*(\theta);\theta_{-n}) \ \forall\, y_n > \theta_n. \tag{33}$$

Since the first term in the right hand side of condition (33) is increasing in y_n and the second term remains constant with a change in y_n, it follows that by making y_n sufficiently large (say some y^*_n) we get $\sum_{j\in N}\hat{\tau}_j(y^*_n,\theta_{-n}) = y^*_n s_1 + \mathcal{T}(\sigma^*(\theta);\theta_{-n}) > 0$ leading to a violation of feasibility. □

Proof of Proposition 4: Consider any $\Gamma^s = (\Omega, O^s(N,s)) \in \mathcal{C}(N)$ such that $|N| = 3$ and, without loss of generality, assume that $s_1 \geq s_2 \geq s_3$. Consider the profile $\theta \in \Theta^3$ such that $\sigma_j^*(\theta) = j$ for all $j \in N$ and in particular $\theta_1/s_1 = a > \theta_2/s_2 = b > \theta_3/s_3 = c > 0$ and assume that (i) $a > \max\{cs_1/s_2, bs_2/s_3\}$. Since $O_j^s(s) = (n+1)s_j/2 > s_i$ for all $j \in N$, using the function $T_j^C(x_j;\theta_{-j})$ given by (13), we can take $\theta_1^* = s_1 c$, $\theta_2^* = s_2 a$ and $\theta_3^* = s_3 a$. Then using the transfers associated with the minimal relative pivotal mechanism (Definition 9) with $T_j^C(x_j;\theta_{-j})$ given by (13) we get the following:

(1) $\hat{\tau}_1(\theta) = -cs_1(s_1 - s_2) - bs_1 s_2$,
(2) $\hat{\tau}_2(\theta) = as_2(s_1 - s_2)$, and
(3) $\hat{\tau}_3(\theta) = as_3(s_1 - s_2) + bs_2 s_3$.

If $s_1 > s_3$, then $\sum_{j \in N} \hat{\tau}_j(\theta) = (s_1 - s_2)(as_2 - cs_1) + (s_1 - s_3)(as_3 - bs_2) = (s_1 - s_2)s_2[a - (cs_1/s_2)] + (s_1 - s_3)s_3[a - (bs_2/s_3)] > 0$ (due to (i)) and we have a contradiction to feasibility. Hence, for feasibility it is necessary that $s_1 \leq s_3$ implying $s_1 \geq s_2 \geq s_3 \geq s_1$. Hence, $s_1 = s_2 = s_3$.

Consider any $\Gamma^{\bar{S}} = (\Omega, O^{\bar{S}}(N,s)) \in \mathcal{E}(N)$ such that $|N| = 3$ and, without loss of generality, assume that $s_1 \geq s_2 \geq s_3$. Consider the profile $\theta \in \Theta^3$ such that $\sigma_j^*(\theta) = j$ for all $j \in N$ and in particular $\theta_1/s_1 = a > \theta_2/s_2 = b > \theta_3/s_3 = c > 0$. Since $O_j^{\bar{S}}(s) = (s_j + A(s))/2 > s_i$ for all $j \in N$, using the function $T_j^E(x_j;\theta_{-j})$ given by (14), we can take $\theta_1^* = s_1 b$, $\theta_2^* = s_2 a$ and $\theta_3^* = s_3 a$. Then using the transfers associated with the minimal relative pivotal mechanism (Definition 9) with $T_j^E(x_j;\theta_{-j})$ given by (14) we get the following:

(1) $\hat{\tau}_1(\theta) = -s_1 b \left(\frac{s_2+s_3}{2}\right)$,
(2) $\hat{\tau}_2(\theta) = s_2 a \left(\frac{s_1-s_3}{2}\right)$, and
(3) $\hat{\tau}_3(\theta) = s_3 a \left(\frac{s_1-s_2}{2}\right) + s_2 s_3 b$.

If $s_1 > s_3$, then $\sum_{j \in N} \hat{\tau}_j(\theta) = \frac{(a-b)}{2}(s_2 s_1 + s_1 s_3 - 2s_2 s_3) > \frac{(a-b)}{2}(s_2 s_3 + s_1 s_3 - 2s_2 s_3) = \frac{(a-b)s_3(s_1-s_2)}{2} \geq 0$ and we have a contradiction to feasibility. Hence, for feasibility we need $s_1 \leq s_3$ implying $s_1 = s_2 = s_3$. □

Proof of Corollary 1: For any profile $\theta \in \Theta^n$ and $i \in N$, consider the type $\theta_i^* \in \Theta$ such that the function $T_i^{QB}(x_i, \theta_{-i})$ (defined in (16)) takes the maximum value, that is, $T_i^{QB}(\theta_i^*, \theta_{-i}) \geq T_i^{QB}(x_i, \theta_{-i})$ for all $x_i \in \Theta^n$. Let $\bar{r}(\theta_{-i}) = ((\bar{r}_j(\theta_{-i}) = \theta_j)_{j \neq i})$ be the vector of agent specific waiting cost in $N \backslash \{i\}$ and $r_i(\theta_{-i}) = (r_1(\theta_{-i}) = \theta_{(1)}, \ldots r_{n-1}(\theta_{-i}) = \theta_{(n-1)})$ be the permutation of $\bar{r}(\theta_{-i})$ such that $r_1(\theta_{-i}) \geq \ldots \geq r_{n-1}(\theta_{-i})$. We can verify that if n is odd, $\theta_i^* \in \{r_{\frac{n-1}{2}}(\theta_{-i}), r_{\frac{n+1}{2}}(\theta_{-i})\}$ and when n is even, $\theta_i^* = r_{\frac{n}{2}}(\theta_{-i})$. Using the resulting θ_i^* that maximizes the function $T_i^{QB}(x_i, \theta_{-i})$ (defined in (16)), we have the following forms of the relative pivotal mechanisms derived for the even and odd cases separately. If n is odd, then we get the transfer given by $\tau_i^{odd}(\theta) + h_i(\theta_{-i})$ where,

$$\tau_i^{odd}(\theta) = \begin{cases} -\sum\limits_{k \in F_i(\sigma^*(\theta)) | 1 < \sigma_k^*(\theta) \leq \frac{n+1}{2}} \theta_k & \text{if } \sigma_i^*(\theta) < \frac{n+1}{2}, \\ 0 & \text{if } \sigma_i^*(\theta) = \frac{n+1}{2}, \\ \sum\limits_{k \in P_i(\sigma^*(\theta)) | \frac{n+1}{2} \leq \sigma_k^*(\theta) < n} \theta_k & \text{if } \sigma_i^*(\theta) > \frac{n+1}{2}, \end{cases} \tag{34}$$

and if n is even, then we get the transfer given by $\tau_i^{even}(\theta) + h_i(\theta_{-i})$ where,
(35)

$$\tau_i^{even}(\theta) = \begin{cases} -\sum\limits_{k \in F_i(\sigma^*(\theta)) | 1 < \sigma_k^*(\theta) \leq \frac{n}{2}} \theta_k - \frac{\theta_f}{2} & \text{if } \sigma_i^*(\theta) < \frac{n}{2}, \sigma_f^*(\theta) = \frac{n}{2} + 1 \text{ and } n > 2, \\ -\frac{\theta_f}{2} & \text{if } \sigma_i^*(\theta) = \frac{n}{2} \text{ and } \sigma_f^*(\theta) = \frac{n}{2} + 1, \\ \frac{\theta_p}{2} & \text{if } \sigma_i^*(\theta) = \frac{n}{2} + 1 \text{ and } \sigma_p^*(\theta) = \frac{n}{2}, \\ \sum\limits_{k \in P_i(\sigma^*(\theta)) | \frac{n}{2} + 1 \leq \sigma_k^*(\theta) < n} \theta_k + \frac{\theta_p}{2} & \text{if } \sigma_i^*(\theta) > \frac{n}{2} + 1, \sigma_p^*(\theta) = \frac{n}{2} \text{ and } n > 2. \end{cases}$$

Observe that, $\tau_i^{odd}(\theta)$ is a K-pivotal mechanism with $K = \frac{n+1}{2}$ while $\tau_i^{even}(\theta)$ is the simple average of two K-pivotal mechanisms-one with $K = n/2$ and the other with $K = n/2 + 1$. We can then generally express,

$$\bar{\tau}_i^a(\theta) = H_i(\theta_{-i}) + \begin{cases} \tau_i^{(\frac{n+1}{2})}(\theta) + & \text{if } n \text{ is odd}, \\ \frac{1}{2}\tau_i^{(\frac{n}{2})}(\theta) + \frac{1}{2}\tau_i^{(\frac{n}{2}+1)}(\theta) & \text{if } n \text{ is even}. \end{cases} \tag{36}$$

□

Proof of Proposition 5: Given that for any queueing problem $\Omega \in \mathcal{Q}(N)$, the symmetrically balanced VCG mechanism satisfies outcome efficiency, strategyproofness, ICB (ECB) and budget balance, it follows that with $O_i = (n+1)/2$ for all

$i \in N$ (which is the bound associated with ICB(ECB)), the result holds. In particular, for any $\theta \in \Theta^n$, the utility of an agent $i \in N$ associated with the symmetrically balanced VCG mechanism satisfies $U_i(\sigma^*(\theta), \tau_i^{sb}(\theta); \theta_i) \geq -(n+1)/2$. Consider any queueing problem with acceptable utility bounds satisfying the following property: For all $i \in N$, $O_i \geq (n+1)/2$ or equivalently, for each $i \in N$, there exists $\beta_i \geq 0$ such that $O_i = \left(\frac{n+1}{2}\right) + \beta_i$. With the symmetrically balanced VCG mechanism we have that for each $\theta \in \Theta^n$ and each $i \in N$,

$$U_i(\sigma^*(\theta), \tau_i^{sb}(\theta); \theta_i) \geq -\left(\frac{n+1}{2}\right) \geq -\left(\frac{n+1}{2}\right) - \beta_i, \text{ for any } \beta_i \geq 0.$$

Therefore, the symmetrically balanced VCG mechanism also ensures outcome efficiency, strategyproofness and budget balance for any acceptable utility bounds vector $O(N) = (O_1, \ldots, O_n)$ such that $O_i \geq (n+1)/2$ for all $i \in N$. □

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ECONOMIC RESEARCH UNIT, INDIAN STATISTICAL INSTITUTE, KOLKATA, INDIA.
E-mail address: `sreoshi.banerjee@gmail.com`

INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH, BHOPAL, INDIA.
E-mail address: `parikshitde@iiserb.ac.in`

ECONOMICS RESEARCH UNIT, INDIAN STATISTICAL INSTITUTE, KOLKATA, INDIA.
E-mail address: `mmitra@isical.ac.in`